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APPLICATIONS OF THE AXIOM OF CHOICE TO CONSTRUCTIONS IN EUCLIDEAN SPACES

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ABSTRACT. We give a short account of corollaries of the Axiom of Choice, which show the existence of curious configurations or colorings of the Euclidean spaces. The paper is readable even with a minimal knowledge of set theory. A possible new result is also proved: pairwise disjoint flags can be elected on each point of the plane.

1. THE AXIOM OF CHOICE.

The axiom[1] of choice, whose statement is given below, is an important tool for proving results on uncountable sets.

> If $\{A_i : i \in I\}$ is an arbitrary set of nonempty sets, then there is a function f such that $f(i) \in A_i$ for $i \in I$.

The AC has various 'good' (or at least useful) consequences, e.g.,

- each set carries a well ordering (an ordering such that every nonempty subset has a least element),
- cardinal comparison satisfies trichotomy,
- $x^2 = x$ holds for every infinite cardinal x,
- every vector space contains a basis, any two bases have the same cardinality, etc, etc.

On the other hand, there are curious corollaries, as

- there is a nonmeasurable set,
- the Banach–Tarski paradox.

Here we give a reader-friendly, short introduction to results of the latter type.

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2. Mazurkiewicz's theorem

One of the popular corollaries of AC is the following theorem of Mazurkiewicz.

Theorem 2.1. (Mazurkiewicz, 1918) *There is a set $A \subseteq \mathbb{R}^2$ such that $|A \cap L| = 2$ for each straight line L.*

The proof uses the fact that there are $\mathfrak{c}$ (continuum) many lines and each is of cardinality $\mathfrak{c}$.

In order to give a motivation for the proof, it is instructive to pretend that there are countably many lines, each of cardinality $\aleph_0$ (countably infinite): $L_0, L_1, \ldots$.

By mathematical induction on i we choose (0,1, or) 2 points from each L_i, such that

- there will be exactly 2 points chosen from L_i,
- no 3 collinear points are chosen.

This is possible: when arriving at L_i, a finite set B of noncollinear points have been chosen.

If $|B \cap L_i| = 2$, go to next stage.

Otherwise, we have to choose 1 or 2 points from L_i. We cannot choose any point that is on a line connecting some 2 points of B. This excludes finitely many points and L_i is infinite, so choice is possible.

To transform this to a real proof, we need

- a (well) order of the set of the lines (AC),
- a machinery that makes possible recursive selection along a well ordering (theorem on transfinite recursion).

3. Rational distances

Paul Erdős asked: Can $\mathbb{R}^d$ be partitioned into countably many parts, each omitting rational distances?

Trivial for $d = 1$: the relation $x \sim y$ if $x - y \in \mathbb{Q}$ is an equivalence relation, with each equivalence class countable.

Partition $\mathbb{R}$ into countably many parts so that the elements of each equivalence class be in distinct parts.

$d = 2$. Reformulated as a graph problem: if X_2 is the graph on $\mathbb{R}^2$ where x, y are joined iff $d(x, y) \in \mathbb{Q}$, then the chromatic number of X_2 is countable, $\mathrm{Chr}(X_2) \leq \aleph_0$.

Erdős–Hajnal: if X does not contain $K_{2,\aleph_1}$, then $\mathrm{Chr}(X) \leq \aleph_0$.

Suffices to show that X_2 does not contain $K_{2,\aleph_1}$. That is, if $a, b \in \mathbb{R}^2$, then $\{x \in \mathbb{R}^2 : d(x,a), d(x,b) \in \mathbb{Q}\}$ is countable. Indeed, if $p, q \in \mathbb{Q}$, then $\{x \in \mathbb{R}^2 : d(x,a) = p, d(x,b) = q\}$ has at most 2 elements. As there are countably many pairs of rational numbers, we are finished.

Erdős and Hajnal prove more: there is a well ordering $\prec$ of $\mathbb{R}^2$ such that for every $x \in \mathbb{R}^2$ there are only finitely many $y \prec x$ with $d(x, y) \in \mathbb{Q}$.

This immediately implies $\mathrm{Chr}(X_2) \le \aleph_0$.

In one of his problem-paper, Erdős claimed that a similar argument works for $d \ge 3$: the corresponding X_d (rational distance graph on $\mathbb{R}^d$) does not contain $K_{d,\aleph_1}$ which implies $\mathrm{Chr}(X_d) \le \aleph_0$.

This is false: there is $K_{3,\aleph_1}$ in $\mathbb{R}^3$: A is formed of 3 points of a line L, B is a circle around L whose plane is perpendicular to L.

Theorem 3.1. (Komjáth, [4]) $\mathrm{Chr}(X_d) = \aleph_0$.

The proof is done by analyzing bipartite subgraphs which occur in examples as the above.

Jim Schmerl later gave a very nice alternative proof. It shows the existence of a well ordering $\prec$ of $\mathbb{R}^d$ such that if $x \in \mathbb{R}^d$, then there are no $x_0, x_1, \dots$ with

- $d(x_i, x) \in \mathbb{Q}$,
- $x_i \prec x$,
- $x_i \to x$ (in the Euclidean topology).

This gives the result: for $x \in \mathbb{R}^d$ let $K(x)$ be a rational cube around x such that for no $y \in K(x)$, $y \prec x$, does $d(y,x) \in \mathbb{Q}$ hold.

Then, if $K(x) = K(y)$, then $d(x,y) \notin \mathbb{Q}$. The coloring $x \mapsto K(x)$ is therefore a coloring with countably many colors, so that no color class contains two points with rational distance.

4. Directions

Theorem 4.1. (Roy O. Davies) *If $E_0, E_1, \dots$ are infinitely many distinct directions on the plane, then there is a coloring $f : \mathbb{R}^2 \to \{0, 1, \dots\}$ such that for each n, $f^{-1}(n)$ intersects each line of direction E_n in at most 1 point.*

In fact, more holds: there is a well ordering $\prec$ of $\mathbb{R}^2$ such that for each x only finitely many n occur such that yx is in direction E_n for some $y \prec x$.

We use the latter result to prove the following theorem.

Theorem 4.2. *Let P be a plane in $\mathbb{R}^3$. Then we can erect pairwise disjoint flags $F(p)$ on each $p \in P$. A flag $F(p)$ is a closed segment $A(p)$ perpendicular to P, with p as one endpoint, the other endpoint above P, plus a rectangle $B(p)$, one of whose side is a subsegment of $A(p)$.*

Proof. Choose a sequence

$$a_0 > b_0 > a_1 > b_1 > \cdots$$

of reals converging to 0. Let $E_0, E_1, \dots$ be distinct directions on P, let $\prec$ be a well ordering of P as in Davies's theorem. For $p \in P$ let $n(p)$ be a natural number such that if $q \prec p$, then qp is not in direction $E_{n(p)}, E_{n(p)+1}, E_{n(p)+2}, \dots$. Then the set $A(p) = [0, a_{n(p)}] \times \{p\}$ and $B(p) = [b_{n(p)}, a_{n(p)}] \times [p, y]$ for some y that py is in direction $E_{n(p)}$. Finally, $F(p) = A(p) \cup B(p)$ is the flag erected from p.

We have to show that $F(p) \cap F(q) = \emptyset$ for $p \neq q$.
Clearly $A(p) \cap A(q) = \emptyset$.
Assume $B(p) \cap A(q) \neq \emptyset$.
Then

- $n(q) \leq n(p)$ (as otherwise $A(q)$ is too short to meet $B(p)$),
- the direction of pq is $E_{n(p)}$ (by the definition of $B(p)$).

Both $p \prec q$ and $q \prec p$ contradict these.
Assume that $B(p) \cap B(q) \neq \emptyset$.

By height considerations, $n(p) = n(q)$. Then both $B(p)$ and $B(q)$ are of direction $E_{n(p)}$, so as they meet, pq is also of direction $E_{n(p)}$. Case $p \prec q$ contradicts the choice of $n(q)$, case $q \prec p$ contradicts the choice of $n(p)$. □

5. METHODS

Most proof uses one of the two methods.

1. Use a basis.
2. Induction on cardinality.

This builds on the following strange argument. Assume, for example, that we want to prove that there is a coloring $f : \mathbb{R}^2 \to \mathbb{N}$ omitting rational distances.

Lemma 5.1. *The following are equivalent:*

(1) *There is a coloring $f : \mathbb{R}^2 \to \mathbb{N}$ omitting rational distances.*
(2) *For every $A \subseteq \mathbb{R}^2$ there is a coloring $f : A \to \mathbb{N}$ omitting rational distances.*

Proof. (2) formally contains (1). On the other hand, if (1) holds, that is, there is a good coloring f for $\mathbb{R}^2$, then $f|A$ will be a good coloring for any $A \subseteq \mathbb{R}^2$. □

And so we can prove the result for each $A \subseteq \mathbb{R}^2$ by transfinite induction on $|A|$.

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