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THE PERIODIC INTEGER ORBITS OF POLYNOMIAL RECURSIONS WITH INTEGER COEFFICIENTS

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Dedicated to Mustafa Kulenović on the occasion of his 70th birthday

ABSTRACT. We show that polynomial recursions $x_{n+1} = x_n^m - k$ where k,m are integers and m is positive have no nontrivial periodic integer orbits for $m \geq 3$. If $m = 2$ then we show that the recursion has integer two-cycles for infinitely many values of k but no higher period orbits. We further show that these statements are true for all quadratic recursions and comment on possible higher order extensions.

1. INTRODUCTION

The quadratic recursion

$$x_{n+1} = x_n^2 - c \tag{1.1}$$

and its topological conjugate, the discrete logistic equation are well-known examples of nonlinear dynamical systems in the sets of real numbers and complex numbers. Starting with an initial value x_0 with $n = 0$ in (1.1) we may calculate the values of x_1, x_2, etc recursively and generate a sequence x_n that is known as a (forward) *orbit or solution* of (1.1).

If the parameter c is a real or complex number, then (1.1) can have a wide variety of bounded orbits. For example, if $c = 2$ then (1.1) has real periodic orbits (or cycles) of all possible periods in the interval $[-2,2]$ depending on the initial value x_0 as well as certain bounded, oscillating but non-periodic orbits that are called *chaotic*; see, e.g. [1], [3].

In this paper, we consider the integer orbits of the more general equation

$$x_{n+1} = x_n^m - k \tag{1.2}$$

Note that if k is an integer then each initial value x_0 in $\mathbb{Z}$ generates an orbit in $\mathbb{Z}$. Some numerical experimentation shows that such integer orbits are typically unbounded. So the question arises as to whether *all* integer orbits are unbounded (except for possible integer fixed points).

We prove that the answer is yes if $m > 2$. But if $m = 2$ then we show that periodic integer orbits with period 2 exist for infinitely many values of k in $\mathbb{Z}$. Our

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results are based on elementary arguments and do no require specialized algebraic methods or concepts.

In addition, we extend the results for $m = 2$ to the general quadratic recursion

$$x_{n+1} = ax_n^2 + bx_n + c,$$

where a, b, c are integers ($a \neq 0$) and determine the equations for the fixed points and the two-cycles in terms of the coefficients a, b, c. We show that the solutions for the general case are qualitatively the same as those for the special case due to topolotical conjugacy. We also determine the cycles of the general case in terms of the given parameters a, b, c to complete the study of the integer solutions of the quadratic recursion.

We close the paper with remarks about extending the results on the quadratic equation to higher order cases as a potential topic for further research.

2. Non-existence of nontrivial periodic orbits if $m > 2$

We begin by recalling a few basic concepts and identifying some exceptional and/or trivial cases. The recursion in (1.2) can be written as

$$x_{n+1} = f(x_n), \quad f(x) = x^m - k.$$

A fixed point of the function $f(x)$, i.e. a solution of the equation

$$f(x) = x$$

is also called a *constant solution* or a *fixed point* of the recursion (1.2). We also call it a *trivial orbit*.

A *periodic orbit* or *cycle* of (1.2) is a sequence $r_0, r_1, r_2, \ldots$ where there is a positive integer p such that

$$r_{n+p} = r_n \quad \text{for all } n. \tag{2.1}$$

If p is the *smallest* positive integer for which (2.1) is true then p is the *period* of r_n; we also call a cycle of period p a *p-cycle* for short. Finally, a *bounded orbit* of (1.2) is a bounded sequence $r_0, r_1, r_2, \ldots$ that satisfies (1.2).

Note that if $m = 1$ then the recursion

$$x_{n+1} = x_n - k \tag{2.2}$$

has the general solution

$$x_n = x_0 - nk.$$

From this, we conclude that every solution of (2.2) diverges to ∞ if $k < 0$ and to $-\infty$ if $k > 0$. If $k = 0$ then every solution of (2.2) is constant (every initial value is fixed). So in the rest of the paper we assume that $m \geq 2$.

If $k = 0$ then (1.2) has two integer fixed points 0 and 1 if m is even and three integer fixed points 0, ± 1 if m is odd.

If $k \neq 0$ and m is odd then using Descartes rule of signs and the intermediate value theorem we see that (1.2) has only one real fixed point γ that is positive if $k > 0$ (i.e. $k \geq 1$) and negative if $k < 0$ (i.e. $k \leq -1$). Further, it is easy to see that all (real) orbits of (1.2) diverge to ∞ if $x_0 > \gamma$ and to $-\infty$ if $x_0 < \gamma$. In particular, (1.2) has no nontrivial, bounded integer orbits if m is odd.

What about possible integer fixed points? An integer j satisfies $x = x^m - k$ if and only if

$$k = j^m - j = j(j^{m-1} - 1).$$

It follows that for every integer j the equation

$$x_{n+1} = x_n^m - j(j^{m-1} - 1)$$

has a fixed point at j. In particular, *(1.2) has integer fixed points for infinitely many values of k.*

For example, if $m = 3$ and k is the (even) number

$$k = j(j^2 - 1) = j(j-1)(j+1),$$

then (1.2) has a fixed point at $x = j$. Similarly, if $m = 4$ and

$$k = j(j^3 - 1) = j(j-1)(j^2 + j + 1), \tag{2.3}$$

then (1.2) has a fixed point at $x = j$.

If $k = 1$ then we may check that (1.2) has an integer cycle of period 2: $-1, 0, -1, 0, \ldots$ for *every* even value of m. Further, the initial value $x_0 = 1$ leads to this cycle in one step. On the other hand, if $|x_0| \geq 2$ then for all $m \geq 2$

$$\begin{aligned} x_1 &\geq 2^m - 1 \geq 3, \\ x_2 &\geq 3^m - 1 \geq 8, \\ &\vdots \end{aligned}$$

which is an increasing sequence of integers that rapidly diverges to ∞. It follows that all orbits with $x_0 \neq -1, 0, 1$ are unbounded when $k = 1$.

We now consider the remaining cases.

The function $f(x) = x^m - k$ has a minimum at 0 and two (real) fixed points α and β where

$$\alpha < 0 < \beta,$$

and

$$\alpha^m = \alpha + k, \qquad \beta^m = \beta + k. \tag{2.4}$$

Note that $k > 0$ if and only if $\beta > 1$ by the right hand side equation above. Further,

$$k = \beta^m - \beta = \beta(\beta^{m-1} - 1),$$

so $k > \beta$ if $\beta > 2^{1/(m-1)}$. In particular, *if* $\beta \geq 2$ *then* $k > \beta$.

The next result shows that it is only necessary to consider orbits of (1.2) that start in $[-\beta,\beta]$ even though this interval is usually not invariant.

Lemma 2.1. *For each $m \geq 2$, if $|x_0| > \beta > 1$, then the orbit generated by (1.2) is unbounded, eventually increasing to ∞.*

Proof. First, note that $f(x) > x$ for all $x > \beta$ so that if $|x_0| > \beta$ then

$$x_1 = f(x_0) > x_0.$$

Further,

$$\begin{aligned} x_1 &= x_0^m - k > \beta^m - k = \beta, \\ x_2 &= x_1^m - k > \beta^m - k = \beta, \\ &\vdots \end{aligned}$$

so by induction, $x_n > \beta$ for every $n \geq 1$. Now,

$$\begin{aligned} x_n - x_{n-1} &= x_{n-1}^m - x_{n-2}^m \\ &= (x_{n-1} - x_{n-2}) \sum_{i=1}^{m} x_{n-1}^{m-i} x_{n-2}^{i-1} \\ &> (x_{n-1} - x_{n-2}) \sum_{i=1}^{m} \beta^{m-i} \beta^{i-1} \\ &= m\beta^{m-1}(x_{n-1} - x_{n-2}). \end{aligned}$$

Doing the same calculation for $x_{n-1} - x_{n-2}$, then for $x_{n-2} - x_{n-3}$ and so on, we obtain by induction

$$x_n - x_{n-1} > (m\beta^{m-1})^{n-1}(x_1 - x_0).$$

Therefore,

$$\begin{aligned} x_n &= x_0 + \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &> x_0 + \sum_{i=1}^{n} (m\beta^{m-1})^{i-1}(x_1 - x_0) \\ &= x_0 + (x_1 - x_0) \frac{(m\beta^{m-1})^n - 1}{m\beta^{m-1} - 1}. \end{aligned}$$

Due to the occurrence of the n-th power of $m\beta^{m-1} > m$ in the last quantity it follows that $x_n \to \infty$ (exponentially fast) as $n \to \infty$ if $x_0 \notin [-\beta,\beta]$. $\square$

It is worth to mention that $m\beta^{m-1} = f'(\beta)$ is the slope of the tangent line to the graph of $f(x)$ at $x = \beta$. We could use this tangent line for an alternative proof but that was not necessary here.

Also notice that the number β is considerably smaller than k; for instance, for $\beta \leq 2$ the right hand side equation in (2.4) gives

$$k = \beta(\beta^{m-1} - 1) \leq 2(2^{m-1} - 1) = 2^m - 2.$$

Next, if $m = 4$ and $k = 2$ then by (2.3) and the intermediate value theorem $\alpha = -1$ and $1 < \beta < 2$. So $k > \beta$ and the interval $[-\beta, \beta] \subset [-2, 2]$ contains the 3 integers $0, \pm 1$. A quick calculation shows that if $x_0 = \pm 1$ then $x_1 = -1$ which is the fixed point, and further, if $x_0 = 0$ then $x_1 = -2 < -\beta$ so $x_n \to \infty$ as $n \to \infty$. So with $k = 2$ the only periodic integer orbit of (1.2) is the trivial one $x_n = -1$ for all n. There is one more bounded integer orbit, namely the one that starts at $x_0 = 1$. Quick calculations show that (1.2) has precisely one integer periodic orbit if $k = 1$, and two fixed points if $k = 0$. Further, (1.2) has no bounded solutions (integer or not) if $k < 0$ (i.e. $k \leq -1$) because

$$x^m - k \geq x^m + 1 > x$$

for all even integers m and all real values of x. We discuss the integer values $k \geq 3$ in the sequel.

Considering orbits that start in $[-\beta, \beta]$, due to the y-axis symmetry we need only check the integers in the interval $[0, \beta]$. If $x_0 \in [0, \beta]$ then

$$|x_1| = |x_0^m - k| \leq \beta$$

if and only if

$$-\beta + k \leq x_0^m \leq \beta + k = \beta^m.$$

Only the left hand side inequality poses a new restriction, namely,

$$x_0 \geq (k - \beta)^{1/m}.$$

Let

$$\gamma = (k - \beta)^{1/m},$$

and note that

$$x_0 \in [\gamma, \beta] \Longrightarrow x_1 \in [-\beta, \beta]$$
$$x_0 \in [0, \gamma) \Longrightarrow x_1 < -\beta.$$

Thus it is necessary that the interval $[\gamma, \beta]$ contain an integer. In this regard, the next lemma is important.

Lemma 2.2. *If m is even and larger than 2, then the length of the interval $[\gamma, \beta]$ is less than 1. Further, $\beta - \gamma \to 0$ for each such value of m as $k \to \infty$.*

Proof. Observe that

$$\beta^m - \gamma^m = k + \beta - (k - \beta) = 2\beta,$$

which yields

$$\beta - \gamma = \frac{2\beta}{\sum_{i=1}^{m} \beta^{m-i}\gamma^{i-1}} < \frac{2}{\beta^{m-2}}.$$

Recall that $\beta > k^{1/m}$ so
$$\beta - \gamma < \frac{2}{k^{(m-2)/m}}. \tag{2.5}$$
With $m > 2$ the right hand side of the above inequality is less than 1 if
$$k > 2^{m/(m-2)}.$$
The largest value of $m/(m-2) = 1 + 2/(m-2)$ occurs at the smallest value of m, i.e. $m = 4$. Thus
$$2^{1+2/(m-2)} \leq 4 \quad \text{for } m = 4, 6, 8, \ldots \; .$$
So if $k \geq 4$ then the right hand side of (2.5) is less than 1 for all $m = 4, 6, 8, \ldots$ and we obtain
$$\beta - \gamma < 1.$$
Further, for each fixed value of m (2.5) implies that $\beta - \gamma \to 0$ as $k \to \infty$. □

The above lemma in particular implies that $[\gamma, \beta]$ contains at most one integer.

Theorem 2.1. *The equation in (1.2) has no nontrivial integer periodic orbits for $m \geq 3$.*

Proof. We discussed the non-existence for all odd values of m earlier, so now assume that m is even and also for this theorem, $m \geq 4$.

We first show that if $x_0 \in (\gamma, \beta)$ then *a non-constant periodic orbit may exist only if* $x_1 \in (-\beta, -\gamma) \cup (\gamma, \beta)$.

Note that since the x-intercept of $f(x) = x^m - k$ is $k^{1/m} \in [\gamma, \beta]$ and f is increasing for $x > 0$ it follows that f maps $[k^{1/m}, \beta]$ one-to-one onto $[0, \beta]$ with $f(k^{1/m}) = 0$. Similarly, f maps the interval $[\gamma, k^{1/m}]$ onto $[-\beta, 0]$ with $f(\gamma) = -\beta$. Since $x_0 \in (\gamma, \beta)$ it follows that $x_1 \in (-\beta, \beta)$.

In order that x_0 and x_1 be part of a periodic orbit it is necessary that $x_2 = f(x_1) \in (-\beta, \beta)$ also. This is possible only if $x_1 \in (-\beta, -\gamma) \cup (\gamma, \beta)$ in which case $|x_1| \in (\gamma, \beta)$.

Notice that *an integer orbit of (1.2) cannot have a period greater than 2* because the set $(-\beta, -\gamma) \cup (\gamma, \beta)$ contains at most two integers.

If x_0, x_1 form an integer orbit of period 2 for (1.2) with $x_0 \in (\gamma, \beta)$, then x_1 must be in the interval $(-\beta, -\gamma)$. It follows that
$$x_1 = -x_0. \tag{2.6}$$

We also require that $x_2 = x_0$ to close the cycle. Therefore,
$$x_0 = x_2 = x_1^m - k = (-x_0)^m - k = x_1, \tag{2.7}$$
where the last equality holds since m is even. The equalities (2.6) and (2.7) hold simultaneously if and only if $x_0 = 0$ which contradicts our assumption about where x_0 is. Therefore, there can be no orbits of period 2 for (1.2).

We have shown that if $k \geq 4$ then the only possible integer cycles of (1.2) are the fixed points. We still need to examine the values of $k < 4$, i.e., $k \leq 3$. We have already checked the solutions of (1.2) for $k \leq 2$. Now, if $k = 3$ then $\beta < 2$ since k is an increasing function of β for $\beta \geq 1$ and at $\beta = 2$

$$k = 2^m - 2 \geq 2^4 - 2 = 14.$$

With $\beta < 2$ the only integers in $[-\beta, \beta]$ are 0 and ± 1. With $k = 3$, if $x_0 = 0, \pm 1$ then $x_1 \leq -2 < -\beta$ so $x_n \to \infty$ as $n \to \infty$. Further, the fixed points of $x^m - 3$ are the zeros of $x^m - x - 3$ which by the intermediate value theorem are in the intervals $(-2, -1)$ and $(1, 2)$ for all $m = 2, 4, 6, \ldots$ Since these fixed points are not integers it follows that (1.2) has no periodic integer orbits with $k = 3$. This completes the proof of the theorem. □

If $m = 2$ then some of the steps in the above argument are invalid; in fact, for $m = 2$ it is the case that $\beta - \gamma \geq 1$ for all k. This opens the way for the existence of 2-cycles.

3. Periodic orbits of the quadratic recursion

Many of the results of the previous section hold when $m = 2$ but as the next lemma shows, it is no longer the case that $\beta - \gamma < 1$. In the case $m = 2$ the fixed points can be determined explicitly by solving the fixed point equation $f(x) = x$. This is the quadratic equation $x^2 - x - k = 0$ whose positive solution is

$$\beta = \frac{1 + \sqrt{1 + 4k}}{2}. \tag{3.1}$$

This in turn gives an explicit formula for $\gamma = \sqrt{k - \beta}$; note that β is an increasing function of k and a simple calculation shows the same to be true for γ. Further, if β is an integer then so is the other fixed point $\alpha = 1 - \beta$. So, unlike the higher degree cases, *integer fixed points always occur in pairs when* $m = 2$.

Lemma 3.1. *For all* $k \geq 2$

$$1 < \beta - \gamma \leq 2. \tag{3.2}$$

Further, the difference $\beta - \gamma$ *is decreasing as a function of* k *with* $\lim_{k \to \infty} (\beta - \gamma) = 1$.

Proof. Note that $\beta - \gamma \leq 2$ if and only if $\gamma \geq \beta - 2$ and this inequality is true if and only if

$$k - \beta = \gamma^2 \geq (\beta - 2)^2 = \beta^2 - 4\beta + 4.$$

Since β is a fixed point, $\beta^2 = k + \beta$ so the above inequality is true if and only if

$$-\beta \geq -3\beta + 4$$

which is true if and only if $\beta \geq 2$. By (3.1) this is the case if $k \geq 2$.

Similarly, $1 < \beta - \gamma$ if and only if

$$k - \beta = \gamma^2 < (\beta - 1)^2 = \beta^2 - 2\beta + 1 = k - \beta + 1,$$

which is obviously true.

The decreasing nature of $\beta-\gamma$ as a function of k may be using derivatives. Next, we take the limit:

$$\lim_{k\to\infty}(\beta-\gamma) = \lim_{k\to\infty}\left(\frac{1+\sqrt{4k+1}}{2} - \sqrt{k-\frac{1+\sqrt{4k+1}}{2}}\right)$$
$$= \frac{1}{2} + \lim_{k\to\infty}\left(\sqrt{k+1/4} - \sqrt{k-1/2-\sqrt{k+1/4}}\right).$$

To calculate the limit of the indeterminate form we multiply and divide by its conjugate to get:

$$\lim_{k\to\infty}(\beta-\gamma) = \frac{1}{2} + \lim_{k\to\infty}\frac{3/4+\sqrt{k+1/4}}{\sqrt{k+1/4}+\sqrt{k-1/2-\sqrt{k+1/4}}}.$$

The limit may now be determined as follows:

$$\lim_{k\to\infty}(\beta-\gamma) = \frac{1}{2} + \lim_{k\to\infty}\frac{3/[4\sqrt{k+1/4}]+1}{1+\sqrt{(k-1/2)/(k+1/4)-\sqrt{k+1/4}/(k+1/4)}}$$
$$= \frac{1}{2} + \frac{1}{1+\sqrt{1-0}} = 1.$$

This concludes the proof. □

It is also useful to write (3.2) as follows:

$$\beta - 2 \le \gamma < \beta - 1. \tag{3.3}$$

Because the length of $[\gamma,\beta]$ is larger than 1 it contains at least one integer for every $k \ge 2$.

We now show that for certain values of k the interval $[\gamma,\beta]$ contains two distinct integers. For the exceptional value $k=2$ we have $\gamma=0$ and $\beta=2$ so $[\gamma,\beta]=[0,2]$ contains three distinct integers.

Lemma 3.2. *Assume that $k \ge 2$ (so that γ is real).*

(a) If $k = j(j+1)$ or $k = j(j+1)+1$ for some positive integer j, then

$$j,\ j+1 \in [\gamma,\beta]. \tag{3.4}$$

(b) If $k \ne j(j+1),\ j(j+1)+1$ for all positive integers j, then $[\gamma,\beta]$ contains exactly one positive integer that is different from both γ and β.

Proof. (a) Note that if $k = j(j+1)$ for some integer j then

$$\beta = \frac{1+\sqrt{1+4j(j+1)}}{2} = \frac{1+2j+1}{2} = j+1,$$

and

$$\gamma = \sqrt{k-\beta} = \sqrt{j^2-1} < j,$$

so (3.4) is true for $k = j(j+1)$. Next, for $k = j(j+1)+1 = j^2+j+1$

$$\beta = \frac{1+\sqrt{1+4j^2+4j+4}}{2} = \frac{1+\sqrt{(1+2j)^2+4}}{2} > j+1,$$

so

$$\gamma = \sqrt{j^2+j+1-\beta} < \sqrt{j^2+j+1-(j+1)} = j.$$

It follows that (3.4) is true for $k = j(j+1)+1$ also and the proof of (a) is complete.

(b) Let β_k be the fixed point of $f(x) = x^2 - k$ and define $\gamma_k = \sqrt{k-\beta_k}$. For all non-negative j define

$$k_j = j(j+1).$$

Note that the sequence of (even) integers k_j is increasing as a function of j and

$$k_{j+1} = (j+1)(j+2) = k_j + 2j + 2.$$

Therefore, for each fixed value of j

$$k_j < j(j+1)+i < k_{j+1}, \qquad i = 1,2,\ldots,2j+1.$$

By Part (a) we know that $[\gamma_{k_j+1}, \beta_{k_j+1}]$ contains both j and $j+1$. Further, since β_k increases with k and the smallest value of k where $\beta_k = j+2$ is $k_{j+1} = (j+1)(j+2)$, it follows that $j+2 \notin [\gamma_{k_j+i}, \beta_{k_j+i}]$ for $i = 1,2,\ldots,2j+1$.
Now, we show that if $i \geq 2$ then $[\gamma_{k_j+i}, \beta_{k_j+i}]$ contains only one integer $j+1$.
To prove this claim, first note that $j+1 < \beta_{k_j+i}$ for all i because β_k is an increasing function of k. Similarly, γ_k increases with k and $\gamma_{k_j+2j+1} < j+1$ for $i = 2j+1$ because after squaring it we obtain

$$k_j + 2j + 1 - \frac{1+\sqrt{1+4(k_j+2j+1)}}{2} < (j+1)^2.$$

Substituting for k_j and doing a little algebra we see that this inequality holds if and only if

$$2j-1 < \sqrt{(2j+1)^2+8j+4},$$

which is obviously true. So $j+1 \in [\gamma_{k_j+i}, \beta_{k_j+i}]$ for $i = 2,\ldots,2j+1$. On the other hand, for $i = 2$ we have $\gamma_{k_j+2} > j$ if and only if

$$k_j + 2 - \frac{1+\sqrt{1+4(k_j+2)}}{2} > j^2,$$

and this inequality holds if and only if

$$2j+3 > \sqrt{(2j+3)^2-8j}.$$

Since the last inequality is true for $j \geq 1$ our claim is justified. Further, γ_k increases with k which implies that $j \notin [\gamma_{k_j+i}, \beta_{k_j+i}]$ for $i = 2,\ldots,2j+1$ and the proof is complete. $\square$

Figure 1 illustrates the above lemma. The upper curve is β_k and the lower is γ_k. The dashed curve shows $\beta_k - 1$. The special values of k where the interval $[c_k, b_k]$ contains two points are highlighted by dots and by numbers in larger font.

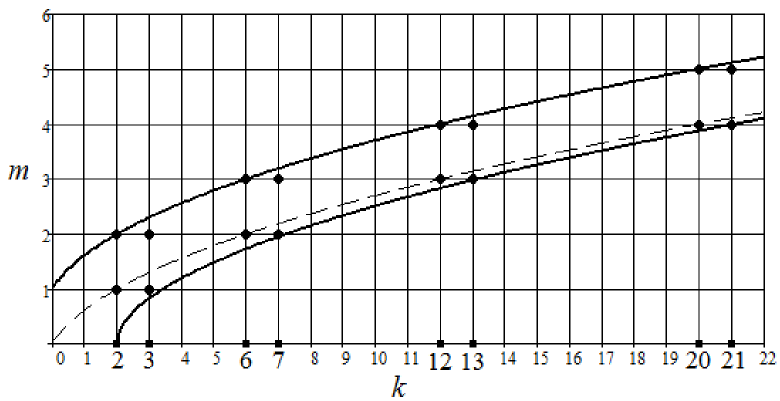

FIGURE 1. *Bounding curves for integer solutions*

Theorem 3.1. *Every sequence r_n of integers that is an orbit of (1.2) for $m = 2$ must satisfy one of the following conditions:*

(a) If $k = j(j+1)$ for some integer j, then r_n is one of two constant sequences, $r_n = j+1$ or $r_n = -j$;

(b) If $k = j(j+1)+1$ for some integer j, then r_n is the 2-cycle $-(j+1)$, j for $n \geq 1$;

(c) If the value of k is not as given in (a) or (b), then r_n diverges to infinity. In particular, there are no integer p-cycles for $p > 2$.

Proof. (a) This was established in Lemma 3.2.

(b) This follows from Lemma 3.2(a) and the observation that

$$f(\pm j) = j^2 - [j(j+1)+1] = -j - 1 = -(j+1),$$
$$f(\pm(j+1)) = (j+1)^2 - [j(j+1)+1] = j.$$

Notice that the orbit $j \to -(j+1) \to j \to -(j+1) \to \cdots$ is the 2-cycle and each of the remaining two points in the set $[-\beta_{k+1}, -\gamma_{k+1}] \cup [\gamma_{k+1}, \beta_{k+1}]$ is mapped to either j or $-(j+1)$ with $k = j(j+1)$.

(c) If the value of k is not as given in (a) or (b) above, i.e. if $k \neq j(j+1), j(j+1)+1$ then by Lemma 3.2(b) the set $[-\beta_k, -\gamma_k] \cup [\gamma_k, \beta_k]$ contains only two points, say, $j \in [\gamma_k, \beta_k]$ and thus $-j \in [-\beta_k, -\gamma_k]$ with $j \neq \gamma_k, \beta_k$ (therefore, j is not a fixed point of f). Further, $f(j) = j^2 - k = -j$ if and only if $k = j(j+1)$ which is ruled out by assumption. Thus $f(j) \neq -j$ which means that $f(j)$ is not in the set $[-\beta_k, -\gamma_k] \cup [\gamma_k, \beta_k]$. Thus, by Lemma 2.1 there are no bounded solutions in this case and therefore, no cycles either. $\square$

4. Extension to the general quadratic map

In this section, we discuss how to extend the results of the previous section to the general quadratic function

$$Q(x) = ax^2 + bx + c \qquad a, b, c \in \mathbb{Z} \quad a \neq 0.$$

In this case $Q : \mathbb{Z} \to \mathbb{Z}$ is a mapping of the integers and the recursion

$$x_{n+1} = Q(x_n) = ax_n^2 + bx_n + c, \quad x_0 \in \mathbb{Z} \tag{4.1}$$

generates integer sequences.

The key observation about Q is that unlike polynomials of degree 3 or greater, the general quadratic function Q is conjugate to the special case

$$f(x) = x^2 - q,$$

where q is a rational number. The only difference between this mapping and the one we studied in the previous section is that q is not an integer if b is odd. Many of the results of the previous section apply to rational q as well so we simply need to point out how to make the connection. We start with the following lemma.

Lemma 4.1. *Let a_i, b_i, c_i for $i = 1, 2$ be fixed real numbers. If $a_1, a_2 \neq 0$ and*

$$a_1(b_1 + c_1) = a_2(b_2 + c_2), \tag{4.2}$$

then the mappings $f_i(x) = a_i(x + b_i)^2 + c_i$, $i = 1, 2$ are topologically conjugate; that is, there is a homeomorphism h such that

$$h \circ f_1 = f_2 \circ h. \tag{4.3}$$

In fact,

$$h(x) = \frac{a_1}{a_2}x + \frac{a_1 b_1 - a_2 b_2}{a_2}. \tag{4.4}$$

Proof. Consider the function $h(x) = \alpha x + \beta$ which is a homeomorphism of the set of real numbers if $\alpha \neq 0$. The equality in (4.3) holds if and only if

$$\alpha a_1(x + b_1)^2 + \alpha c_1 + \beta = a_2(\alpha x + \beta + b_2)^2 + c_2,$$

i.e.,

$$\alpha a_1(x + b_1)^2 + \alpha c_1 + \beta = a_2\alpha^2 \left(x + \frac{b_2 + \beta}{\alpha}\right)^2 + c_2.$$

The last equality holds if α, β can be chosen so that

$$\alpha a_1 = a_2\alpha^2, \quad b_1 = \frac{b_2 + \beta}{\alpha}, \quad \alpha c_1 + \beta = c_2. \tag{4.5}$$

The first two of the above equalities gives

$$\alpha = \frac{a_1}{a_2}, \quad \beta = \frac{a_1 b_1 - a_2 b_2}{a_2}.$$

Further, α, β must satisfiy the third equality in (4.5)

$$\alpha c_1 + \beta = c_2,$$

i.e.,

$$a_1 c_1 + a_1 b_1 - a_2 b_2 = a_2 c_2.$$

The last equality is equivalent to (4.2). □

Next, observe that since

$$Q(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

the following corollary of Lemma 4.1 is obtained by setting

$$a_1 = a,\ b_1 = \frac{b}{2a},\ c_1 = c - \frac{b^2}{4a} \quad \text{and} \quad a_2 = 1,\ b_2 = 0$$

in (4.2) and using the conjugate map h.

Lemma 4.2. *The quadratic function $Q(x)$ is topologically conjugate to the translation*

$$f(x) = x^2 - q, \quad q = \frac{b^2}{4} - \frac{b}{2} - ac = \frac{b(b-2)}{4} - ac. \tag{4.6}$$

Every orbit r_n of (4.6) uniquely corresponds via the homeomorphism h to an orbit s_n of (4.1) as follows

$$r_n = as_n + \frac{b}{2}.$$

Equivalently,

$$s_n = \frac{r_n}{a} - \frac{b}{2a}. \tag{4.7}$$

The main issue now is to show that there are rational orbits r_n of

$$x_{n+1} = x_n^2 - q \tag{4.8}$$

that yield all the integer orbits s_n of (4.1) via (4.7).

We begin with the observation that if b is even, then q in (4.6) is an integer so we may apply Theorem 3.1 directly to the quadratic function $f(x)$ in (4.6) and obtain the next corollary about the integer orbits of (4.1).

Corollary 4.1. *Assume that b is an even integer in (4.1).*

(a) There is at least one integer fixed point for (4.1) if

$$\frac{b(b-2)}{4} - ac = j(j+1)$$

for some integer j. The integer fixed point is one of the following (both of them if $a = \pm 1$)

$$\frac{j}{a} - \frac{b-2}{2a}, \quad -\frac{j}{a} - \frac{b}{2a}.$$

(b) There is an integer 2-cycle for (4.1) if

$$\frac{b(b-2)}{4} - ac = j(j+1)+1$$

for some integer j. This 2-cycle is

$$\frac{j}{a} - \frac{b}{2a}, \quad -\frac{j}{a} - \frac{b+2}{2a}.$$

(c) If ac is not as given in (a) or (b), then every integer orbit of (4.1) increases to ∞ if $a > 0$ or decreases to $-\infty$ if $a < 0$. In particular, (4.1) has no integer p-cycles if $p \geq 3$.

Note that in the special case where $a = 1$ and $b = 0$ the above corollary reduces to Theorem 3.1 (with $k = -c$).

To illustrate Corollary 4.1 with an example let l be any positive integer and consider

$$x_{n+1} = x_n^2 + 2x_n - l, \tag{4.9}$$

where $a = 1$, $b = 2$ and $c = -l$. The recursion in (4.9) has a pair of integer fixed points j and $-j-1$ if $l = j(j+1)$ and it has an integer 2-cycle $j-1$, $-j-2$ if $l = j(j+1)+1$. There are no other cycles of (4.9) for any value of l.

For the equation

$$x_{n+1} = -2x_n^2 + 2x_n + 1$$

we have $a = -2$, $b = 2$ and $c = 1$. With $ac = -2$, Part (a) of Corollary 4.1 holds if $j = 1$; of the two fixed points

$$\frac{j}{a} - \frac{b-2}{2a} = -\frac{1}{2}, \quad -\frac{j}{a} - \frac{b}{2a} = 1$$

only one is an integer. There are no proper cycles in this case.

If b is *odd* then Theorem 3.1 is not applicable but a modified form of Corollary 4.1 holds. The key observation is the following:

If a, b, c have integer values in (4.6) then $4q$ is an integer.

In fact,

$$4q = b^2 - 2b - 4ac = (b-1)^2 - 1 - 4ac.$$

From this equality we obtain

$$1 + 4q = (b-1)^2 - 4ac,$$

which is the discriminant of the fixed point equations for both Q and its conjugate f. Indeed, the fixed points of Q are the solutions of $Q(x) = x$, i.e.,

$$ax^2 + (b-1)x + c = 0,$$

which yields

$$\frac{-(b-1) \pm \sqrt{(b-1)^2 - 4ac}}{2a} \tag{4.10}$$

while from $f(x) = x$, i.e.,

$$x^2 - x - q = 0$$

we obtain

$$\frac{1 \pm \sqrt{1+4q}}{2}. \tag{4.11}$$

In order that the numbers in (4.10) and (4.11) be rational it is necessary that under the square roots we have perfect squares.

Now, suppose that b is odd. Then from (4.10) we obtain integers if and only if $(b-1)^2 - 4ac$ is the square of an *even* integer, i.e.,

$$(b-1)^2 - 4ac = (2m)^2.$$

The last equation may be written as

$$ac = \left(\frac{b-1}{2}\right)^2 - m^2. \tag{4.12}$$

Thus, when b is odd $Q(x)$ has integer fixed points if the product ac is a number of the above type for some integer m. This is how (4.12) modifies Part (a) of Corollary 4.1 when b is odd. For example, the quadratic recursion

$$x_{n+1} = x_n^2 + x_n - 1, \tag{4.13}$$

with $a = b = 1$ and $c = -1$ gives

$$\left(\frac{b-1}{2}\right)^2 - 1 = -1 = ac,$$

so $m = 1$. Indeed, (4.13) has a pair of integer fixed points ± 1.

To extend these observations to cycles with lengths larger than 1 we consider $f(x) = x^2 - q$ and the fixed points in (4.11). Using notation analogous to what we previsously discussed for the case of integer q, define

$$B_q = \frac{1+\sqrt{1+4q}}{2}, \qquad C_q = \sqrt{q - B_q}.$$

These are the same as the earlier parameters b_k and c_k. In fact, if we think of k (or q) as *real* numbers then they are indeed the same functions but now we check their values for rational q. Notice that

$$q = \frac{b^2}{4} - \frac{b}{2} + \frac{1}{4} - ac - \frac{1}{4} = \left(\frac{b-1}{2}\right)^2 - ac - \frac{1}{4}.$$

Looking back at Figure 1, *when b is odd, we check the region between the two curves at integer values less 1/4* on the horizontal axis; that is, at $k - 1/4$ rather than at integers k.

With this in mind, if we set $q = k - 1/4$ (or $1+4q = k$) in the square root in B_q, then we obtain

$$\sqrt{1+4q} = \sqrt{1+4\left(k-\frac{1}{4}\right)} = 2\sqrt{k},$$

which is rational (in fact, integer) if and only if $k = j^2$ is a perfect square. This gives

$$B_q = \frac{1+2\sqrt{k}}{2} = \frac{1}{2} + j.$$

Therefore, if $q = j^2 - 1/4$ where j is an integer, then *the fixed point B_q has an integer value plus 1/2*. Figure 2 illustrates this relationship.

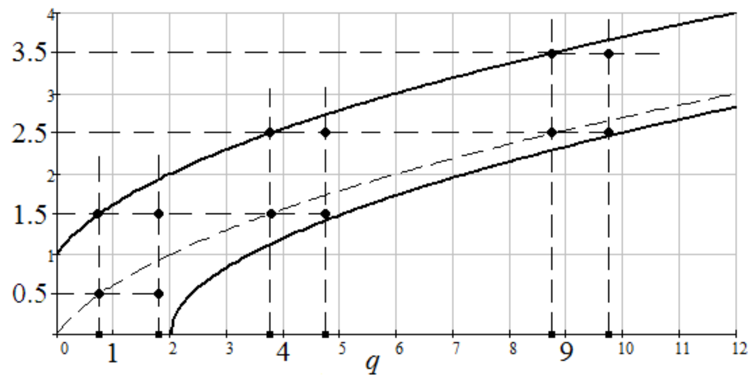

FIGURE 2. *Bounding curves, odd b*

In Figure 2 we see that for the "consecutive" values $j^2 - 1/4$ and $j^2 - 1/4 + 1$ a square of side 1 fits in the region between the curves B_q and C_q just like the earlier case where q was integer. The second fixed point of $Q(x)$ is

$$A_q = \frac{1}{2} - j,$$

since $A_q B_q = -q$. Note that A_q is in the mirror image of $[C_q, B_q]$, i.e. the interval $[-B_q, C_q]$. Its negative $-A_q = j - 1/2$ is in $[C_q, B_q]$ and this is the other point that we see directly below B_q in Figure 2.

Earlier, in Figure 1 we saw that the 2-cycles occurred at the value of k next to the one that produced the fixed points. A similar situation appears in Figure 2; the values $q = j^2 - 1/4 + 1$ for $q = 19/4$ ($j = 2$) and $q = 39/4$ ($j = 3$) are shown. These are the 2-cycle candidates and we need only verify this.

Note that the top points at the numbers $q = j^2 - 1/4 + 1$ are $\alpha = j + 1/2$. If we set $\beta = f(\alpha)$, then

$$\beta = \left(j+\frac{1}{2}\right)^2 - \left(j^2+\frac{3}{4}\right) = j - \frac{1}{2},$$

and

$$f(\beta) = \left(j - \frac{1}{2}\right)^2 - \left(j^2 + \frac{3}{4}\right) = -\left(j + \frac{1}{2}\right) = -\alpha.$$

Since $f(\alpha) = f(-\alpha) = \beta$ and $f(\beta) = f(-\beta) = -\alpha$ we see that $-\alpha, \beta, -\alpha, \beta, \ldots$ is indeed a 2-cycle in the set

$$[-B_q, C_q] \cup [C_q, B_q]. \tag{4.14}$$

This gives us the proper modification of Part (b) Corollary 4.1 when b is odd. Finally, the occurrence of p-cycles for $p \geq 3$ is prohibited because it is impossible to fit enough "integer plus half" points in the set in (4.14) for each value of q.

We summarize these facts in the following.

Corollary 4.2. *Assume that b is an odd integer in (4.1).*

(a) *There is at least one integer fixed point for (4.1) if*

$$\left(\frac{b-1}{2}\right)^2 - ac = j^2$$

for some integer j. The integer fixed point is one of the following (both if $a = \pm 1$)

$$\frac{j}{a} - \frac{b-1}{2a}, \quad -\frac{j}{a} - \frac{b-1}{2a}.$$

(b) *There is an integer 2-cycle for (4.1) if*

$$\left(\frac{b-1}{2}\right)^2 - ac = j^2 + 1$$

for some integer j. This 2-cycle is

$$-\frac{j}{a} - \frac{b+1}{2a}, \quad \frac{j}{a} - \frac{b+1}{2a}.$$

(c) *If ac is not as given in (a) or (b), then every integer orbit of (4.1) increases to ∞ if $a > 0$ or decreases to $-\infty$ if $a < 0$. In particular, (4.1) has no integer p-cycles if $p \geq 3$.*

For instance, consider the quadratic equation

$$x_{n+1} = x_n^2 + x_n - 2, \tag{4.15}$$

with $a = b = 1$ and $c = -2$ satisfies the conditions of the above corollary with $j = 1$ since

$$\left(\frac{b-1}{2}\right)^2 - ac = 2 = j^2 + 1.$$

So there is a 2-cycle whose points are

$$-\frac{j}{a} - \frac{b+1}{2a} = -1 - 1 = -2, \quad \frac{j}{a} - \frac{b+1}{2a} = 1 - 1 = 0.$$

The periodic integer solution of (4.15) is $-2, 0, -2, 0, \ldots$ which can be easily verified by direct substitution into (4.15).

5. Remarks on the higher order quadratic recursions

The results in the preceding two sections determine the solutions or orbits of the first-order quadratic recursion. A natural question at this stage is what can be expected of the integer solutions when delays are present in the recursion, i.e. when the quadratic equation has order greater than 1.

The behavior of the integer solutions of higher order quadratic recursion are unsurprisinly more complex and display a greater variety of cases. The proper investigation of these solutions may be the subject of another paper.

In this section, we comment on some of the special cases as a way of alerting the reader about the possiblities. While there seems to be no significant published results on the integer solutions of higher order equations, the *real* solutions of quadratic polynomial equations of higher order have been investigated in substantial detail; see, e.g. [2].

First, consider the straightforward case of the basic higher order quadratic recursion

$$x_{n+1} = x_{n-m}^2 - k, \tag{5.1}$$

where m is a positive integer and the initial values $x_0, x_1, \ldots, x_m$ and k are integers (note that the difference equation (5.1) has order $m+1$). Theorem 3.1 can be readily extended to this case.

If $k = j(j+1)$ for some positive integer j then by Theorem 3.1 the first order recursion (1.2) has two fixed points or constant solutions: $-j$ and $j+1$. Setting some of the $m+1$ initial values equal to one of these and setting the remaining initial values to the other we obtain a solution of period $m+1$ for (5.1), i.e. an $(m+1)$-cycle.

Most of these cycles represent the same cycle that is shifted forward or backward. For example, let $j = 1$. Then $j(j+1) = 2$ and for $m = 2$ we obtain the recursion of order 3

$$x_{n+1} = x_{n-2}^2 - 2. \tag{5.2}$$

This equation has two integer fixed points $-j = -1$ and $j+1 = 2$ which also serve as constant integer solutions. The selection

$$x_0 = x_1 = -1, \quad x_2 = 2$$

of initial values generates the 3-cycle

$$-1, -1, 2, -1, -1, 2, -1, -1, 2, \ldots,$$

which is the same as the cycle

$$-1, 2, -1, -1, 2, -1, -1, 2, -1, \ldots$$

generated by the selection

$$x_0 = -1, \quad x_1 = 2, \quad x_2 = -1$$

but shifted one step to the right.

One the other hand, the selection

$$x_0 = -1, \quad x_1 = x_2 = 2$$

does generate a different 3-cycle

$$-1, 2, 2, -1, 2, 2, \ldots \,.$$

If $k = j(j+1)+1$, then according to Theorem 3.1 the first-order equation has an integer solution of period 2 that is given by the numbers $-(j+1)$ and j. This cycle generates corresponding cycles (not all distinct) for the higher order equations.

For example, with $j = 1$ the third-order equation

$$x_{n+1} = x_{n-2}^2 - 3 \tag{5.3}$$

has solutions of period 6 such as

$$-2, -2, -2, 1, 1, 1, -2, -2, -2, 1, 1, 1, -2, -2, -2, 1, 1, 1, \ldots,$$

corresponding to the selection

$$x_0 = x_1 = x_2 = -2.$$

Some selections of initial values generate shorter cycles. For instance, the selection

$$x_0 = 1, \quad x_1 = -2, \quad x_2 = 1$$

generates the sequence

$$1, -2, 1, -2, 1, -2, 1, -2, 1, 1, -2, 1, \ldots$$

that has period 2.

The general quadratic equation corresponds to many different higher order equations that may be expressed collectively as

$$x_{n+1} = a x_{n-m}^2 + b x_{n-k} + c. \tag{5.4}$$

In this form, the larger of m and k gives the order of (5.4), as either $m+1$ or $k+1$.

We have a complete description of solutions for the first-order case where $m = k = 0$. These results may be readily extended to (5.4) if

$$m = k,$$

using Corollaries 4.1 and 4.2 similarly to the discussion above for the basic higher order case.

If $m \neq k$, then new results are required for finding the periodic integer solutions of (5.4) in $\mathbb{Z}^2$.

Perhaps a good place to start is with the second-order equations where $m+k = 1$. There is a greater variety of solutions in these cases than in the case $m = k = 1$. For example, we may verify that the recursion

$$x_{n+1} = x_n^2 - x_{n-1} - 1,$$

with initial values $x_0 = -1$ and $x_1 = 0$ has a 3-cycle

$$-1, 0, 0, -1, 0, 0, -1, 0, 0, \dots,$$

whereas the same equation with initial values $x_0 = -1$ and $x_1 = 1$ has a 4-cycle

$$-1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, \dots \, .$$

These cycles are different than the ones that we might get for the case $m = k = 1$ and suggest a greater level of complexity.

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