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EQUIENERGETIC GRAPHS USING CARTESIAN PRODUCT AND GENERALIZED COMPOSITION

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ABSTRACT. The energy of a graph is the sum of the absolute values of its eigenvalues. Two graphs of same order are said to be equienergetic if they have same energy. Several papers dealing with equienergetic graphs exists in the literature and most of these papers consists of equieneregtic regular graphs. In this paper we give regular as well as non-regular, equieneregtic graphs using the Cartesian product and also by generalized composition through equitable partition.

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. A graph is an *r-regular graph* if all its vertices have the same degree equal to r. The *complement* of a graph G is the graph $\overline{G}$ with vertex set $V(\overline{G}) = V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* of G, denoted by $L(G)$ is the graph whose vertex set has one-to-one correspondence with the edges of G and two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. For $k = 1, 2, \ldots$, the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$. Let K_n be the complete graph on n vertices and $K_{p,q}$ be the complete bipartite graph on $n = p+q$ vertices [11].

The *adjacency matrix* of a graph G is an $n \times n$ matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The *characteristic polynomial* of G, denoted by $\phi(G : x)$ is the characteristic polynomial of $A(G)$. The eigenvalues of $A(G)$ labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are called the *eigenvalues* of G and their collection is called the *spectrum* of G [7]. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of G with respective multiplicities $m_1, m_2, \ldots, m_k$, then the spectrum of G is denoted by

$$\mathrm{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix},$$

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where $m_1 + m_2 + \cdots + m_k = n$.

Two graphs are said to be *cospectral* if they have the same spectra.

The *energy* of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of G [9]. That is,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

More results on graph energy can be found in [10, 14, 18].

Two graphs G_1 and G_2 of the same order are said to be *equienergetic*, if they have the same energy. Numerous results on non-cospectral, equienergetic graphs have appeared in the literature. Balakrishnan [3] and Stevanović [30] constructed equienergetic graphs using tensor product of graphs. Ramane et al. [22] and Liu et al. [15] constructed equienergetic graphs by joining two graphs. Bonifácio et al. [4] and Ramane et al. [21] obtained some classes of equienergetic graphs through the Cartesian product, tensor product and strong product of graphs. Ramane et al. [24, 25] obtained non-cospectral equienergetic iterated line graphs from regular graphs. For other results on equienergetic graphs one can see [1,2,5,8,12,15,16,19,20,29].

Most of the papers mentioned above are on equienergetic graphs, which are regular. In this paper we obtain equieneregtic graphs using the Cartesian product. Further we construct equienergetic graphs using generalized composition through equitable partition. Using these techniques we can get both regular and non-regular equienergetic graphs.

Theorem 1.1. [27] *Let G be an r-regular graph on n vertices and m edges, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigenvalues of $L(G)$ are $\lambda_i + r - 2$, $i = 1, 2, \ldots, n$ and -2 ($m - n$ times).*

Theorem 1.2. [26] *Let G be an r-regular graph of order n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigenvalues of $\overline{G}$ are $n - r - 1, -\lambda_2 - 1, \ldots, -\lambda_n - 1$.*

Theorem 1.3. [25] *Let G_1 and G_2 be regular graphs of the same order n and of the same degree $r \geq 3$. Then for $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$ are equienergetic.*

Theorem 1.4. [20] *Let G_1 and G_2 be regular graphs of the same order n and of the same degree $r \geq 3$. Then for $k \geq 2$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic.*

2. Equienergetic graphs using Cartesian product

The *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2.

Theorem 2.1. [6] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G_1 and $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of G_2, then the eigenvalues of $G_1 \Box G_2$ are $\lambda_i + \mu_j$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$.*

Theorem 2.2. *Let $L^k(G)$ denote the k-th iterated line graph of G for $k = 0, 1, 2, \dots$. Then*

(a) $\mathcal{E}(L^k(K_{n,n} \square K_{n-1})) = \mathcal{E}(L^k(K_{n-1,n-1} \square K_n))$ *for all* $n \geq 5$.

(b) $\mathcal{E}(\overline{L^k(K_{n,n} \square K_{n-1})}) = \mathcal{E}(\overline{L^k(K_{n-1,n-1} \square K_n)})$ *for all* $n \geq 4$.

Proof. As $K_{n,n} \square K_{n-1}$ and $K_{n-1,n-1} \square K_n$ are both regular graphs of the same order and of the same degree, by Theorems 1.3 and 1.4, the results are true for $k \geq 2$. Now, it is enough to prove the statement for $k = 0, 1$.

(a) Case 1.1: When $k = 0$.

$$\operatorname{Spec}(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\operatorname{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

Therefore by Theorem 2.1

$$\operatorname{Spec}(K_{n,n} \square K_{n-1}) = \begin{pmatrix} 2n-2 & n-1 & n-2 & -1 & -2 & -n-1 \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 \end{pmatrix}. \quad (2.1)$$

Similarly

$$\operatorname{Spec}(K_{n-1,n-1} \square K_n) = \begin{pmatrix} 2n-2 & n-2 & n-1 & -1 & 0 & -n \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 \end{pmatrix}. \quad (2.2)$$

Therefore

$$\begin{aligned} \mathcal{E}(K_{n,n} \square K_{n-1}) &= |2n-2| + |n-1|(n-2) + |n-2|(2n-2) \\ &\quad + |-1|(2n-2)(n-2) + |-2| + |-n-1|(n-2) \\ &= 2(n-1)(3n-4) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}(K_{n-1,n-1} \square K_n) &= |2n-2| + |n-2|(n-1) + |n-1|(2n-4) \\ &\quad + |-1|(2n-4)(n-1) + |-n|(n-1) \\ &= 2(n-1)(3n-4). \end{aligned}$$

Hence $\mathcal{E}(K_{n,n} \square K_{n-1}) = \mathcal{E}(K_{n-1,n-1} \square K_n)$.

Case 1.2: When $k = 1$.

Both $K_{n,n} \square K_{n-1}$ and $K_{n-1,n-1} \square K_n$ are regular graphs of the same order $2n(n-1)$ and of the same degree $2n-2$. Hence by Theorem 1.1 and by Eqs. (2.1) and (2.2),

$$\operatorname{Spec}(L(K_{n,n} \square K_{n-1})) = \begin{pmatrix} 4n-6 & 3n-5 & 3n-6 & 2n-5 & 2n-6 & n-5 & -2 \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 & 2n(n-1)(n-2) \end{pmatrix} \quad (2.3)$$

and

$$\operatorname{Spec}(L(K_{n-1,n-1}\square K_n)) = \begin{pmatrix} 4n-6 & 3n-6 & 3n-5 & 2n-5 \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) \end{pmatrix.$$
$$\begin{matrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & 2n(n-1)(n-2) \end{pmatrix}. \quad (2.4)$$

Therefore

$$\begin{aligned}\mathcal{E}(L(K_{n,n}\square K_{n-1})) &= |4n-6|+|3n-5|(n-2)+|3n-6|(2n-2) \\ &\quad +|2n-5|(2n-2)(n-2)+|2n-6|+|n-5|(n-2) \\ &\quad +|-2|2n(n-1)(n-2) \\ &= 8n(n-1)(n-2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}(L(K_{n,n}\square K_{n-1})) &= |4n-6|+|3n-6|(n-1)+|3n-5|(2n-4) \\ &\quad +|2n-5|(2n-4)(n-1)+|2n-4|+|n-4|(n-1) \\ &\quad +|-2|2n(n-1)(n-2) \\ &= 8n(n-1)(n-2).\end{aligned}$$

Hence $\mathcal{E}(L(K_{n,n}\square K_{n-1})) = \mathcal{E}(L(K_{n-1,n-1}\square K_n))$.

(b) Case 2.1: When $k=0$.
Graphs $K_{n,n}\square K_{n-1}$ and $K_{n-1,n-1}\square K_n$ are regular graphs of order $2n(n-1)$ and of degree $2n-2$. By Theorem 1.2 and by Eqs. (2.1) and (2.2),

$$\operatorname{Spec}\left(\overline{K_{n,n}\square K_{n-1}}\right) = \begin{pmatrix} 2n^2-4n+1 & -n & -n+1 & 0 & 1 & n \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 \end{pmatrix}$$

and

$$\operatorname{Spec}\left(\overline{K_{n-1,n-1}\square K_n}\right) = \begin{pmatrix} 2n^2-4n+1 & -n+1 & -n & 0 & -1 & -1+n \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 \end{pmatrix}.$$

Therefore

$$\begin{aligned}\mathcal{E}\left(\overline{K_{n,n}\square K_{n-1}}\right) &= |2n^2-4n+1|+|-n|(n-2)+|-n+1|(2n-2) \\ &\quad +1+|n|(n-2) \\ &= 2(3n^2-6n+2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}\left(\overline{K_{n-1,n-1}\square K_n}\right) &= |2n^2-4n+1|+|-n+1|(n-1)+|-n|(2n-4)\\ &\quad +|-1|+|-1+n|(n-1)\\ &= 2(3n^2-6n+2).\end{aligned}$$

Hence $\mathcal{E}\left(\overline{K_{n,n}\square K_{n-1}}\right)=\mathcal{E}\left(\overline{K_{n-1,n-1}\square K_n}\right)$.

Case 2.2: When $k=1$.
Graphs $L(K_{n,n}\square K_{n-1})$ and $L(K_{n-1,n-1}\square K_n)$ are regular graphs of order $2n^3-4n^2+2n$ and of degree $4n-6$. By Theorem 1.2 and by Eqs. (2.3) and (2.4),

$$\operatorname{Spec}\left(\overline{L(K_{n,n}\square K_{n-1})}\right)=\begin{pmatrix}2n^3-4n^2-2n+5 & -3n+4 & -3n+5 & -2n+4 & -2n+5 & -n+4 & 1\\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 & 2n(n-1)(n-2)\end{pmatrix}$$

and

$$\operatorname{Spec}\left(\overline{L(K_{n-1,n-1}\square K_n)}\right)=\begin{pmatrix}2n^3-4n^2-2n+5 & -3n+5 & -3n+4 & -2n+4 & -2n+3 & -n+3 & 1\\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 & 2n(n-1)(n-2)\end{pmatrix}.$$

Therefore

$$\begin{aligned}\mathcal{E}\left(\overline{L(K_{n,n}\square K_{n-1})}\right) &= |2n^3-4n^2-2n+5|+|-3n+4|(n-2)\\ &\quad +|-3n+5|(2n-2)+|-2n+4|(2n-2)(n-2)\\ &\quad +|-2n+5|+|-n+4|(n-2)+2n(n-1)(n-2)\\ &= 2(4n^3-10n^2+2n+5)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}\left(\overline{L(K_{n,n}\square K_{n-1})}\right) &= |2n^3-4n^2-2n+5|+|-3n+5|(n-1)\\ &\quad +|-3n+4|(2n-4)+|-2n+4|(2n-4)(n-1)\\ &\quad +|-2n+3|+|-n+3|(n-1)+2n(n-1)(n-2)\\ &= 2(4n^3-10n^2+2n+5).\end{aligned}$$

Hence $\mathcal{E}\left(\overline{L(K_{n,n}\square K_{n-1})}\right)=\mathcal{E}\left(\overline{L(K_{n-1,n-1}\square K_n)}\right)$. □

Theorem 2.3. *Let G_1 and G_2 be r-regular, equienergetic graphs of order n. Then for $p\geq r$, $\mathcal{E}\left(\overline{G_1\square K_p}\right)=\mathcal{E}\left(\overline{G_2\square K_p}\right)$.*

Proof.

$$\mathrm{Spec}(K_p) = \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}.$$

Let

$$\mathrm{Spec}(G_1) = \begin{pmatrix} r & \lambda_2 & \dots & \lambda_k \\ 1 & m_2 & \dots & m_k \end{pmatrix},$$

where $1+\sum_{i=2}^{k} m_i = n$.

By Theorem 2.1

$$\mathrm{Spec}(G_1 \square K_p) = \begin{pmatrix} r+p-1 & r-1 & \lambda_2+p-1 & \cdots & \lambda_k+p-1 & \lambda_2-1 & \cdots & \lambda_k-1 \\ 1 & p-1 & m_2 & \cdots & m_k & m_2(p-1) & \cdots & m_k(p-1) \end{pmatrix}.$$

The graph $G_1 \square K_p$ is a regular graph on np vertices with regularity $r+p-1$. Therefore by applying Theorem 1.2 to the spectrum of $G_1 \square K_p$, we get

$$\mathrm{Spec}\left(\overline{G_1 \square K_p}\right) = \begin{pmatrix} np-r-p & -r & -\lambda_2-p & \cdots & -\lambda_k-p & -\lambda_2 & \cdots & -\lambda_k \\ 1 & p-1 & m_2 & \cdots & m_k & m_2(p-1) & \cdots & m_k(p-1) \end{pmatrix}.$$

Therefore

$$\begin{aligned} \mathcal{E}\left(\overline{G_1 \square K_p}\right) &= |np-r-p| + |-r|(p-1) + \sum_{i=2}^{k} m_i|-\lambda_i-p| + \sum_{i=2}^{k} m_i(p-1)|-\lambda_i| \\ &= np-r-p+r(p-1) + \sum_{i=2}^{k} m_i(\lambda_i+p) \\ &\quad + (p-1)\sum_{i=2}^{k} m_i|\lambda_i|, \quad \text{since } \lambda_i+p \geq -r+p \geq 0 \\ &= np-r-p+r(p-1) + [-r+p(n-1)] + (p-1)[\mathcal{E}(G_1)-r] \\ &= 2(np-p-r) + (p-1)\mathcal{E}(G_1). \end{aligned} \tag{2.5}$$

Similarly we can show that

$$\mathcal{E}\left(\overline{G_2 \square K_p}\right) = 2(np-p-r) + (p-1)\mathcal{E}(G_2). \tag{2.6}$$

Thus result follows from Eqs. (2.5) and (2.6) as G_1 and G_2 are equienergetic. □

Theorem 2.4. *Let G_1 and G_2 be r-regular, equienergetic graphs of order n. Then for $p \geq r$, $\mathcal{E}(G_1 \square K_{p,p}) = \mathcal{E}(G_2 \square K_{p,p})$.*

Proof.

$$\mathrm{Spec}(K_{p,p}) = \begin{pmatrix} p & 0 & -p \\ 1 & 2p-2 & 1 \end{pmatrix}.$$

Let

$$\mathrm{Spec}(G_1) = \begin{pmatrix} r & \lambda_2 & \dots & \lambda_k \\ 1 & m_2 & \dots & m_k \end{pmatrix},$$

where $1+\sum_{i=2}^{k} m_i = n$.

By Theorem 2.1

$$\mathrm{Spec}(G_1 \Box K_{p,p}) = \begin{pmatrix} r+p & r & r-p & \lambda_2+p & \cdots & \lambda_k+p & \lambda_2 & \cdots & \lambda_k & \lambda_2-p & \cdots & \lambda_k-p \\ 1 & 2(p-1) & 1 & m_2 & \cdots & m_k & 2m_2(p-1) & \cdots & 2m_k(p-1) & m_2 & \cdots & m_k \end{pmatrix}.$$

Therefore

$$\begin{aligned} \mathcal{E}(G_1 \Box K_{p,p}) &= |r+p| + |r|2(p-1) + |r-p| + \sum_{i=2}^{k} m_i|\lambda_i + p| \\ &+ \sum_{i=2}^{k} 2m_i(p-1)|\lambda_i| + \sum_{i=2}^{k} m_i|\lambda_i - p| \\ &= r+p+2r(p-1)+p-r+\sum_{i=2}^{k} m_i(\lambda_i+p) \\ &+ 2(p-1)\sum_{i=2}^{k} m_i|\lambda_i| + \sum_{i=2}^{k} m_i(p-\lambda_i) \\ &\quad \text{since } \lambda_i + p \geq -r+p \geq 0 \text{ and } \lambda_i - p \leq r-p \leq 0 \\ &= 2np + 2(p-1)\mathcal{E}(G_1). \end{aligned} \tag{2.7}$$

Similarly we can show that

$$\mathcal{E}(G_2 \Box K_{p,p}) = 2np + 2(p-1)\mathcal{E}(G_2). \tag{2.8}$$

Thus the result follows from Eqs. (2.7) and (2.8) as G_1 and G_2 are equienergetic. □

3. Equienergetic graphs using generalized composition

If G is a graph with vertices $v_1, v_2, \dots, v_n$ then the graph $G[H_1, H_2, \dots, H_n]$ called *generalized composition*, is formed by taking the disjoint graphs $H_1, H_2, \dots, H_n$ and then joining every point of H_i to every point of H_j whenever the vertices v_i and v_j are adjacent in G [28].

The automorphism group of G induces a partition of its points into orbits. If the vertices u and v are in the same orbit, say O_i, then certainly for any orbit O_j, the vertex u must have as many neighbors in O_j as v does. Thus, denoting the set of points adjacent to u as $N(u)$, we have $\mid N(u) \cup O_j \mid = \mid N(v) \cup O_j \mid$. A partition

$V_1 \cup V_2 \cup \ldots \cup V_k$ of $V(G)$ is *equitable* or (*coloration*) [17] if for each i and for all $u, v \in V_i$, $\mid N(u) \cap V_j \mid = \mid N(v) \cap V_j \mid$ for all j [28].

For example, the partition of $V(G)$ into singletons is always equitable. In generalized composition, if a graph H is regular then $V(H)$ can be taken as a partite set in an equitable partition. If P is an equitable partition, we associate with it a $k \times k$ matrix $Q = [q_{ij}]$, where $q_{ij} = \mid N(v) \cap V_j \mid$ for any $v \in V_i$. Such a matrix is called a *quotient matrix* or a *coloration matrix*.

Let $\phi(M : x)$ denotes the characteristic polynomial of the matrix M.

Theorem 3.1. *[28] If $V_1, V_2, \ldots, V_k$ is an equitable partition of a graph G, then $\phi(Q : x)$ divides $\phi(G : x)$.*

Theorem 3.2. *[28] Let G be a graph on n vertices. If H_i is an r_i-regular garph, $i = 1, 2, \ldots, n$, then $V(H_1) \cup V(H_2) \cup \ldots \cup V(H_n)$ is an equitable partition of $G[H_1, H_2, \ldots, H_n]$ and*

$$\phi(G[H_1, H_2, \ldots, H_n] : x) = \phi(Q : x) \prod_{i=1}^{n} \frac{\phi(H_i : x)}{x - r_i}.$$

Theorem 3.3. *[7] Let K_n be a complete graph of order n and H_i be a regular graph of order k_i and of degree r_i, such that $m_i - r_i = p$ for $i = 1, 2, \ldots, n$. Then $K_n[H_1, H_2, \ldots, H_n]$ is $(t = s - p)$-regular with order $s = \sum_{i=1}^{n} k_i$ so that*

$$\phi(K_n[H_1, H_2, \ldots, H_n] : x) = (x - t)(x + s - t)^{n-1} \prod_{i=1}^{n} \frac{\phi(H_i : x)}{x - r_i}$$

Proposition 3.1. [22] Let G_1 and G_2 be r-regular, equienergetic graphs of the same order n. Then $K_2[G_1, K_p]$ and $K_2[G_2, K_p]$ are equienergetic.

Proposition 3.2. [23] Let G_1 and G_2 be r-regular, equienergetic graphs of the same order n and H be any regular graph. Then $K_2[G_1, H]$ and $K_2[G_2, H]$ are equienergetic.

Lemma 3.1. *Let G be an r-regular graph of order n and H be a non-regular graph of order k. Then the spectrum of $K_2[G, H]$ is $Spec(Q) \cup (Spec(G) \setminus \{r\})$ where Q is the quotient matrix of $K_2[G, H]$.*

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_k\}$. As G is regular and H is non-regular, the partition $V(G) \cup \left(\cup_{i=1}^{k} \{v_i\}\right)$ can be taken as an equitable partition and the quotient matrix can be written as

$$\begin{pmatrix} r & 1 & 1 & \ldots & 1 \\ n & 0 & a_{12} & \ldots & a_{1k} \\ n & a_{21} & 0 & \ldots & a_{2k} \\ \vdots & & & \vdots & \\ n & a_{k1} & a_{k2} & \ldots & 0 \end{pmatrix}$$

where
$$a_{ij}=\begin{cases}1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } H\\ 0 & \text{otherwise.}\end{cases}$$

Therefore by Theorem 3.1, $\mathrm{Spec}(K_2[G,H]) = \mathrm{Spec}(Q) \cup (\mathrm{Spec}(G) \setminus \{r\})$. □

Theorem 3.4. *Let G be any graph of order n. Let H_{e_1} and H_{e_2} be r-regular, equienergetic graphs of the same order k. If H_i is any graph of order k_i, $i = 1,2,\ldots,n$, then $G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]$ and $G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]$ are equienergetic, where one of H_i is replaced by H_{e_1} and H_{e_2} respectively for $1 \le i \le n$ in $G[H_1,H_2,\ldots,H_i,\ldots,H_n]$. In addition if $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ are equienergetic, then $\overline{G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]}$ and $\overline{G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]}$ are also equienergetic.*

Proof. If H_i is a regular graph, then $V(H_i)$ is a partite set and if H_i is non-regular then each vertex of H_i can be taken as a partite set for $1 \le i \le n$. These two kind of partite sets together form an equitable partition of $G[H_1,H_2,\ldots,H_i,\ldots,H_n]$. The quotient matrices corresponding to the equitable partition of $G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]$ and of the equitable partition of $G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]$ are the same and can be written as
$$Q=\begin{pmatrix} D_1 & a_{12}b_{12}B_{12} & \cdots & a_{1i}b_{1i}B_{1i} & \cdots & a_{1n}b_{1n}B_{1n}\\ a_{21}b_{21}B_{21} & D_2 & \cdots & a_{2i}b_{2i}B_{2i} & \cdots & a_{2n}b_{2n}B_{2n}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{i1}b_{i1}B_{i1} & a_{i2}b_{i2}B_{i2} & \cdots & D_i & \cdots & a_{in}b_{in}B_{in}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{n1}b_{n1}B_{n1} & a_{n2}b_{n2}B_{n2} & \cdots & a_{ni}b_{ni}B_{ni} & \cdots & D_n \end{pmatrix},$$
where
$$a_{ij}=\begin{cases}1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G\\ 0 & \text{otherwise,}\end{cases}$$
$$D_i=\begin{cases}A(H_i) & \text{if } H_i \text{ is a non-regular graph}\\ r_i & \text{if } H_i \text{ is } r_i\text{-regular graph,}\end{cases}$$
$$b_{ij}=\begin{cases}k_j & \text{if } H_i \text{ is non-regular and } H_j \text{ is regular}\\ k_j & \text{if } H_i \text{ and } H_j \text{ are both regular}\\ 1 & \text{if } H_i \text{ is regular and } H_j \text{ is non-regular}\\ 1 & \text{if } H_i \text{ and } H_j \text{ are both non-regular}\end{cases}$$
and
$$B_{ij}=\begin{cases}J_{k_i\times 1} & \text{if } H_i \text{ is non-regular and } H_j \text{ is regular}\\ 1 & \text{if } H_i \text{ and } H_j \text{ are both regular}\\ J_{1\times k_j} & \text{if } H_i \text{ is regular and } H_j \text{ is non-regular}\\ J_{k_i\times k_j} & \text{if } H_i \text{ and } H_j \text{ are both non-regular,}\end{cases}$$
where J is the matrix whose all entries are equal to 1.

Therefore by Theorem 3.2,

$$\mathrm{Spec}(G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]) = \left(\bigcup_{\substack{j=1 \\ j\neq i \\ \text{whenever } H_j \text{ is } r_j\text{-regular}}}^{n} [\mathrm{Spec}(H_j)\backslash\{r_j\}] \right) \cup (\mathrm{Spec}(H_{e_1})\backslash\{r\}) \cup \mathrm{spec}(Q).$$

Therefore

$$\mathcal{E}\left(G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]\right) = \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } H_j \text{ is } r_j\text{-regular}}}^{n} [\mathcal{E}(H_j)-r_j] \right) + \mathcal{E}(H_{e_1}) - r + \mathcal{E}(Q).$$

Similarly we can get

$$\mathcal{E}\left(G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]\right) = \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } H_j \text{ is } r_j\text{-regular}}}^{n} [\mathcal{E}(H_j)-r_j] \right) + \mathcal{E}(H_{e_2}) - r + \mathcal{E}(Q).$$

Since H_{e_1} and H_{e_2} are equienergetic, the result follows.

In addition if $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ are equienergetic then $\overline{G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]}$ and $\overline{G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]}$ are also equienergetic. The proof follows with the same equitable partition by incorporating the following changes:

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } \overline{G} \\ 0 & \text{otherwise} \end{cases}$$

and

$$D_i = \begin{cases} A(\overline{H_i}) & \text{if } H_i \text{ is a non-regular graph} \\ k_i - r_i - 1 & \text{if } H_i \text{ is } r_i\text{-regular graph.} \end{cases}$$

Hence

$$\mathcal{E}\left(\overline{G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]}\right) = \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } H_j \text{ is } r_j\text{-regular}}}^{n} \left[\mathcal{E}(\overline{H_j}) - (k_j - r_j - 1)\right] \right) + \mathcal{E}(\overline{H_{e_1}}) - (k - r - 1) + \mathcal{E}(Q)$$

and

$$\mathcal{E}\left(\overline{G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]}\right) = \left(\sum_{\substack{j=1\\ j\neq i\\ \text{whenever } H_j \text{ is } r_j\text{-regular}}}^{n} \left[\mathcal{E}(\overline{H_j})-(k_j-r_j-1)\right]\right)+\mathcal{E}(\overline{H_{e_2}})-(k-r-1)+\mathcal{E}(Q).$$

Since $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ are equienergetic, the result follows. □

Corollary 3.1. *Let G be any graph of order n, H_{e_1} and H_{e_2} be r-regular, equienergetic graphs of the same order k. If H_i is an r_i-regular graph of order k_i, $i = 1,2,\ldots,n$, then $G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]$ and $G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]$ are equienergetic, where one of H_i is replaced by H_{e_1} and H_{e_2} respectively for $1 \le i \le n$ in $G[H_1,H_2,\ldots,H_i,\ldots,H_n]$. In addition if $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ are equienergetic then $\overline{G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]}$ and $\overline{G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]}$ are also equienergetic.*

Proof. As $H_1,H_2,\ldots,H_i,\ldots,H_n$ are regular graphs, $\bigcup_{i=1}^{n} V(H_i)$ is an equitable partition of $G[H_1,H_2,\ldots,H_i,\ldots,H_n]$. The quotient matrix corresponding to the equitable partition of $G[H_1,H_2,\ldots,H_i,\ldots,H_n]$ for $H_i = H_{e_1}$ and for $H_i = H_{e_2}$ is same and can be written as

$$Q=\begin{pmatrix} r_1 & a_{12}k_2 & \cdots & a_{1i}k_i & \cdots & a_{1n}k_n\\ a_{21}k_1 & r_2 & \cdots & a_{2i}k_i & \cdots & a_{2n}k_n\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{i1}k_1 & a_{i2}k_2 & \cdots & r_i & \cdots & a_{in}k_n\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{n1}k_1 & a_{n2}k_2 & \cdots & a_{ni}k_i & \cdots & r_n \end{pmatrix},$$

where

$$a_{ij}=\begin{cases}1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G\\ 0 & \text{otherwise.}\end{cases}$$

Therefore by Theorem 3.2,

$$\operatorname{Spec}(G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]) = \bigcup_{\substack{j=1\\ j\neq i}}^{n} [\operatorname{Spec}(H_j)\backslash\{r_j\}]\cup(\operatorname{Spec}(H_{e_1})\backslash\{r\})\cup\operatorname{Spec}(Q).$$

Therefore

$$\mathcal{E}(G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]) = \sum_{\substack{j=1\\ j\neq i}}^{n}[\mathcal{E}(H_j)-r_j]+\mathcal{E}(H_{e_1})-r+\mathcal{E}(Q).$$

Similarly we can get

$$\mathcal{E}\left(G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]\right)=\sum_{\substack{j=1\\ j\neq i}}^{n}[\mathcal{E}(H_j)-r_j]+\mathcal{E}(H_{e_2})-r+\mathcal{E}(Q).$$

Since H_{e_1} and H_{e_2} are equienergetic, the result follows.

In addition if $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ are equienergetic then $\overline{G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]}$ and $\overline{G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]}$ are also equienergetic. The proof follows with the same equitable partition by incorporating the following changes:

$$a_{ij}=\begin{cases}1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } \overline{G}\\ 0 & \text{otherwise,}\end{cases}$$

with k_i-r_i-1 as diagonal entries in the above quotient matrix Q for $1\leq i\leq n$.

Hence

$$\begin{aligned}&\mathcal{E}\left(\overline{G[H_1,H_2,\ldots,H_{e_1},\ldots,H_n]}\right)\\ &\quad=\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mathcal{E}(\overline{H_j})-(k_j-r_j-1)\right]+\mathcal{E}(\overline{H_{e_1}})-(k-r-1)+\mathcal{E}(Q)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{E}\left(\overline{G[H_1,H_2,\ldots,H_{e_2},\ldots,H_n]}\right)\\ &\quad=\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mathcal{E}(\overline{H_j})-(k_j-r_j-1)\right]+\mathcal{E}(\overline{H_{e_2}})-(k-r-1)+\mathcal{E}(Q).\end{aligned}$$

Since $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ are equienergetic, the result follows. □

Corollary 3.2. *Let G be any graph of order n. If $H_{e_{1i}}$ and $H_{e_{2i}}$ are equienergetic graphs of the same order k_i and of the same degree r_i, $i=1,2,\ldots,n$, then $G[H_{e_{11}},H_{e_{12}},\ldots,H_{e_{1n}}]$ and $G[H_{e_{21}},H_{e_{22}},\ldots,H_{e_{2n}}]$ are equienergetic.*

Proof. As $H_{e_{ji}}$ is a regular graph for $j=1,2$ and $1\leq i\leq n$, the partitions $\bigcup_{i=1}^{n}V(H_{e_{1i}})$ and $\bigcup_{i=1}^{n}V(H_{e_{2i}})$ are equitable partitions of $G[H_{e_{11}},H_{e_{12}},\ldots,H_{e_{1n}}]$ and $G[H_{e_{21}},H_{e_{22}},\ldots,H_{e_{2n}}]$ respectively. The quotient matrix corresponding to these equitable partitions is the same and can be written as

$$Q=\begin{pmatrix} r_1 & a_{12}k_2 & \cdots & a_{1i}k_i & \cdots & a_{1n}k_n\\ a_{21}k_1 & r_2 & \cdots & a_{2i}k_i & \cdots & a_{2n}k_n\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{i1}k_1 & a_{i2}k_2 & \cdots & r_i & \cdots & a_{in}k_n\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{n1}k_1 & a_{n2}k_2 & \cdots & a_{ni}k_i & \cdots & r_n\end{pmatrix},$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G \\ 0 & \text{otherwise.} \end{cases}$$

Therefore by Theorem 3.2,

$$\mathrm{Spec}(G[H_{e_{11}}, H_{e_{12}}, \dots, H_{e_{1n}}]) = \bigcup_{i=1}^{n} [\mathrm{Spec}(H_{e_{1i}}) \setminus \{r_i\}] \cup \mathrm{Spec}(Q).$$

Therefore

$$\mathcal{E}\left(G[H_{e_{11}}, H_{e_{12}}, \dots, H_{e_{1n}}]\right) = \sum_{i=1}^{n} [\mathcal{E}(H_{e_{1i}}) - r_i] + \mathcal{E}(Q).$$

Similarly we get

$$\mathcal{E}\left(G[H_{e_{21}}, H_{e_{22}}, \dots, H_{e_{2n}}]\right) = \sum_{i=1}^{n} [\mathcal{E}(H_{e_{2i}}) - r_i] + \mathcal{E}(Q).$$

Since $H_{e_{1i}}$ and $H_{e_{2i}}$ are equienergetic, the result follows. □

Remark 3.1.

(1) In Corollary 3.1, if $G = K_n$ and', $H_1, H_2, \dots, H_i, \dots, H_n$ with $H_i = H_{e_1}$ and $H_i = H_{e_2}$ are such that $k_1 - r_1 = k_2 - r_2 = \cdots = k - r = \cdots = k_n - r_n = p$ and $k_1 + k_2 + \cdots + k + \cdots + k_n = s$, then $G[H_1, H_2, \dots, H_{e_1}, \dots, H_n]$ and $G[H_1, H_2, \dots, H_{e_2}, \dots, H_n]$ are of order s and of degree $t = s - p$ with $\mathcal{E}(Q) = t + (n-1)(s-t)$. This way of construction enables us to construct a family of regular equienergetic graphs.
(2) If the conditions in the above remark are not satisfied, we get a family of non-regular equienergetic graphs.
(3) The proposed method of construction given in Theorem 3.4 leads to a family of co-spectral as well as non co-spectral equienergetic graphs when a pair (H_{e_1}, H_{e_2}) of co-spectral or non co-spectral equienergetic regular graphs of the same order and of the same degree are considered.
(4) Propositions 3.1 and 3.2 are particular cases of Corollary 3.1.

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REFERENCES

[1] C. Adiga and B. R. Rakshith, *On spectra of variants of the corona of two graphs and some new equienergetic graphs*, Discuss. Math. Graph Theory, 36 (2016), 127–140.

[2] A. Ali, S. Elumalai, T. Mansour and M. A. Rostami, *On the complementary equienergetic graphs*, MATCH Commun. Math. Comput. Chem., 83 (2020), 555–570.

[3] R. Balakrishnan, *The energy of a graph*, Linear Algebra Appl., 387 (2004), 287–295.

[4] A. S. Bonifácio, C. T. M. Vinagre and N. M. M. de Abreu, *Constructing pairs of equienergetic and non-cospectral graphs*, Appl. Math. Lett., 21 (2008), 338–341.

[5] V. Brankov, D. Stevanović and I. Gutman, *Equienergetic chemical trees*, J. Serb. Chem. Soc., 69 (2004), 549–553.

[6] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, Berlin, 2012.

[7] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2010.

[8] H. A. Ganie, S. Pirzada and A. Iványi, *Energy, Laplacian energy of double graphs and new families of equienergetic graphs*, Acta Univ. Sapientiae, Informatica, 6 (2014), 89–116.

[9] I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungszentrum Graz, 103 (1978), 1–22.

[10] I. Gutman and H. S. Ramane, *Research on graph energies in 2019*, MATCH Commun. Math. Comput. Chem., 84 (2020), 277–292.

[11] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi, 1998.

[12] G. Indulal and A. Vijayakumar, *On pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem., 55 (2006), 83–90.

[13] G. Indulal and A. Vijayakumar, *Equienergetic self-complementary graphs*, Czechoslovak Math. J., 58 (2008), 911–919.

[14] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[15] J. Liu and B. Liu, *On a pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem., 59 (2008), 275–278.

[16] O. Miljković, B. Furtula, S. Radenković and I. Gutman, *Equienergetic and almost equienergetic trees*, MATCH Commun. Math. Comput. Chem., 61 (2009), 451–461.

[17] D. L. Powers and M. M. Sulaiman, *The walk partition and colorations of a graph*, Linear Algebra Appl., 48 (1982), 145–159.

[18] H. S. Ramane, *Energy of graphs*, in: M. Pal, S. Samanta, A. Pal (Eds.), Handbook of Research on Advanced Applications of Graph Theory in Modern Society, IGI Global, Hershey, 2020, pp. 267–296.

[19] H. S. Ramane, K. Ashoka, B. Parvathalu, D. Patil and I. Gutman, *On complementary equienergetic strongly regular graphs*, Discrete Math. Lett., 4 (2020), 50–55.

[20] H. S. Ramane, I. Gutman, H. B. Walikar and S. B. Halkarni, *Equienergetic complement graphs*, Kragujevac J. Sci., 27 (2005), 67–74.

[21] H. S. Ramane, B. Parvathalu, D. D. Patil and K. Ashoka, *Graphs equienergetic with their complements*, MATCH Commun. Math. Comput. Chem., 82 (2019), 471–480.

[22] H. S. Ramane and H. B. Walikar, *Construction of equienergetic graphs*, MATCH Commun. Math. Comput. Chem., 57 (2007), 203–210.

[23] H. S. Ramane, H. B. Walikar and I. Gutman, *Equienergetic graphs*, J. Combin. Math. Combin. Comput., 69 (2009), 165–173.

[24] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog and I. Gutman, *Equienergetic graphs*, Kragujevac J. Math., 26 (2004), 5–13.

[25] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog and I. Gutman, *Spectra and energies of iterated line graphs of regular graphs*, Appl. Math. Lett., 18 (2005), 679–682.
[26] H. Sachs, *Über selbstkomplementare Graphen*, Publ. Math. Debrecen, 9 (1962), 270–288.
[27] H. Sachs, *Über Teiler, Faktoren und charakteristische Polynome von Graphen*, Teil II. Wiss. Z. TH Ilmenau., 13 (1967), 405–412.
[28] A. J. Schwenk, *Computing the characteristic polynomial of a graph*, Graphs Combin., 406 (1974), 153–172.
[29] I. Stanković, M. Milošević and D. Stevanović, *Small and not so small equienergetic graphs*, MATCH Commun. Math. Comput. Chem., 61 (2009), 443–450.
[30] D. Stevanović, *Energy and NEPS of graphs*, Linear Multilinear Algebra, 53 (2005), 67–74.

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