SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.12.3.14
Vol.12 (25), No.2, (2016), 429–446, Suppl.

ORDER, TYPE AND COTYPE OF GROWTH FOR p-ADIC ENTIRE FUNCTIONS

ALAIN ESCASSUT, KAMAL BOUSSAF AND ABDELBAKI BOUTABAA

Dedicated to the memory of Professor Marc Krasner

Abstract. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entires functions on $\mathbb{K}$. For an $f \in \mathcal{A}(\mathbb{K})$, similarly to complex analysis, one can define the order of growth as $\rho(f) = \limsup_{r\to+\infty} \frac{\log(\log(|f|(r))}{\log r}$. When $\rho(f) \neq 0, +\infty$, one can define the type of growth as $\sigma(f) = \limsup_{r\to+\infty} \frac{\log(|f|(r))}{r^{\rho(f)}}$. But here, we can also define the cotype of growth as $\psi(f) = \limsup_{r\to+\infty} \frac{q(f,r)}{r^{\rho(f)}}$ where $q(f,r)$ is the number of zeros of f in the disk of center 0 and radius r. Then we have $\rho(f)\sigma(f) \leq \psi(f) \leq e\rho(f)\sigma(f)$. Moreover, if ψ or σ are veritable limits, then $\rho(f)\sigma(f) = \psi(f)$ and this relation is conjectured in the general case. Many other properties are examined concerning $\rho(f)$, $\sigma(f)$, $\psi(f)$. Particularly, we prove that if an entire function f has finite order, then $\dfrac{f'}{f^2}$ takes every value infinitely many times and applications are shown to branched values of meromorphic functions.

1. Order and cotype of growth

We denote by $\mathbb{K}$ an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $|\ .\ |$. Analytic functions inside a disk or in the whole field $\mathbb{K}$ were introduced and studied in many books. Given $\alpha \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in \mathbb{K} \mid |x - \alpha| \leq R\}$, by $d(\alpha, R^-)$ the disk $\{x \in \mathbb{K} \mid |x-\alpha| < R\}$, by $C(\alpha, r)$ the circle $\{x \in \mathbb{K} \mid |x-\alpha| = r\}$, by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$

2010 *Mathematics Subject Classification.* 12J25; 30D35; 30G06; 46S10.
Key words and phrases. p-adic entire functions, growth of entire functions, order, type and cotype of growth.
This paper was presented at the International Scientific Conference *Graded structures in algebra and their applications*, dedicated to the memory of Prof. Marc Krasner, IUC-Dubrovnik, Croatia, September, 22-24, 2016.

(i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ (i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$). Given $f \in \mathcal{M}(\mathbb{K})$, we will denote by $q(f,r)$ the number of zeros of f in $d(0,r)$, taking multiplicity into account and by $u(f,r)$ the number of distinct multiple zeros of f in $d(0,r)$. Throughout the paper, log denotes the Neperian logarithm.

Analytic functions in a p-adic field were studied in many documents and particularly by Marc Krasner at the beginning [15], [16]. Here, we will focus on entire functions and examine the notions of order of growth and type of growth for functions of order t as they are shown in complex analysis. We will also introduce a new notion of cotype of growth in relation with the distribution of zeros in disks which plays a major role in processes that are quite different from those in complex analysis. This has an application to the question whether an entire function can be devided by its derivative inside the algebra of entire functions.

Let us shortly recall classical results [8], [9], [10], [14], [15], [16].

Notation. Given $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| \mid |x| = r\}$.

Theorem A. *$|\ .\ |(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Suppose $f(0) \neq 0$ and let $a_1, \dots, a_m$ be the various zeros of f in $d(0,r)$ with $|a_n| \leq |a_{n+1}|$, $1 \leq n \leq m-1$, each zero a_n having a multiplicity order w_n. Then*

$$\log(|f|(r)) = \log(|f(0)|) + \sum_{n=1}^{m} w_n(\log(r) - \log(|a_n|)).$$

Similarly to the definition known on complex entire functions [18], given $f \in \mathcal{A}(\mathbb{K})$, the superior limit

$$\limsup_{r \to +\infty} \Big(\frac{\log(\log(|f|(r)))}{\log(r)} \Big)$$

is called *the order of growth of f* or *the order of f* in brief and is denoted by $\rho(f)$. We say that f has *finite order* if $\rho(f) < +\infty$.

The following Theorems 1, 2, 3, 4, 5, 6, 7 are proven in [4].

Theorem 1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then:*

$$\rho(f+g) \leq \max(\rho(f), \rho(g)),$$
$$\rho(fg) = \max(\rho(f), \rho(g)).$$

Corollary 1.1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then $\rho(f^n) = \rho(f)\ \forall n \in \mathbb{N}^*$. If $\rho(f) > \rho(g)$, then $\rho(f+g) = \rho(f)$.*

Remark. ρ is an ultrametric extended semi-norm.

Notation. Given $t \in [0, +\infty[$, we denote by $\mathcal{A}(\mathbb{K}, t)$ the set of $f \in \mathcal{A}(\mathbb{K})$ such that $\rho(f) \leq t$ and we set

$$\mathcal{A}^0(\mathbb{K}) = \bigcup_{t \in [0,+\infty[} \mathcal{A}(\mathbb{K}, t).$$

Corollary 1.2. *For any $t \geq 0$, $\mathcal{A}(\mathbb{K}, t)$ is a $\mathbb{K}$-subalgebra of $\mathcal{A}(\mathbb{K})$. If $t \leq u$, then $\mathcal{A}(\mathbb{K}, t) \subset \mathcal{A}(\mathbb{K}, u)$ and $\mathcal{A}^0(\mathbb{K})$ is also a $\mathbb{K}$-subalgebra of $\mathcal{A}(\mathbb{K})$.*

Theorem 2. *Let $f \in \mathcal{A}(\mathbb{K})$ and let $P \in \mathbb{K}[x]$. Then $\rho(P \circ f) = \rho(f)$ and $\rho(f \circ P) = \deg(P)\rho(f)$.*

Theorem 3. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be transcendental. If $\rho(f) \neq 0$, then $\rho(f \circ g) = +\infty$. If $\rho(f) = 0$, then $\rho(f \circ g) \geq \rho(g)$.*

Theorem 4. *Let $f \in \mathcal{A}(\mathbb{K})$ be not identically zero. If there exists $s \geq 0$ such that*

$$\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^s} \right) < +\infty$$

then $\rho(f)$ is the lowest bound of the set of $s \in [0, +\infty[$ such that

$$\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^s} \right) = 0.$$

Moreover, if $\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^t} \right)$ is a number $b \in]0, +\infty[$, then $\rho(f) = t$. If there exists no s such that $\limsup_{r \to +\infty} \left(\frac{q(f,r)}{r^s} \right) < +\infty$, then $\rho(f) = +\infty$.

Example. Suppose that for each $r > 0$, we have $q(f,r) \in [r^t \log r, r^t \log r + 1]$. Then of course, for every $s > t$, we have

$$\limsup_{r \to +\infty} \frac{q(f,r)}{r^s} = 0$$

and $\limsup_{r \to +\infty} \frac{q(f,r)}{r^t} = +\infty$, so there exists no $t > 0$ such that $\frac{q(f,r)}{r^t}$ have non-zero superior limit $b < +\infty$.

Definition and notation. Let $t \in [0, +\infty[$ and let $f \in \mathcal{A}(\mathbb{K})$ of order t. We set $\psi(f) = \limsup_{r \to +\infty} \frac{q(f,r)}{r^t}$ and call $\psi(f)$ the *cotype* of f. Moreover, we wet $\widetilde{\psi}(f) = \liminf_{r \to +\infty} \frac{q(f,r)}{r^t}$.

Theorem 5. *Let $f,\ g \in \mathcal{A}^0(\mathbb{K})$ be such that $\rho(f) \geq \rho(g)$. Then $\psi(fg) \leq \psi(f) + \psi(g)$. Moreover, if $\rho(f) = \rho(g)$ then $\max(\psi(f), \psi(g)) \leq \psi(fg)$.*

Proof. Set $\rho(f) = s,\ \rho(g) = t$. By Theorem 1, we have $\rho(f.g) = \rho(f)$. For each $r > 0$, we have $q(f.g, r) = q(f, r) + q(g, r)$ hence

$$\psi(fg) = \limsup_{r\to+\infty} \frac{q(f,r) + q(g,r)}{r^s} \le \limsup_{r\to+\infty} \frac{q(f,r)}{r^s} + \limsup_{r\to+\infty} \frac{q(g,r)}{r^t}$$

hence $\psi(fg) \le \psi(f) + \psi(g)$.

Now, suppose $s = t$. Then

$$\psi(fg) = \limsup_{r\to+\infty} \frac{q(f,r) + q(g,r)}{r^s} \ge \limsup_{r\to+\infty} \frac{\max(q(f,r), q(g,r))}{r^s}$$
$$= \max(\psi(f), \psi(g)),$$

which ends the proof. □

Thanks to Theorem 1 in [1] we can derive the following Theorem 6:

Theorem 6. *Let $f \in \mathcal{A}^0(\mathbb{K})$. Then for every $b \in \mathbb{K}$, $\dfrac{f'}{f^2} - b$ has infinitely many zeros.*

Proof. Let $t = \rho(f)$ and take $c > \psi(f)$. Since $f \in \mathcal{A}^0(\mathbb{K})$, by Theorem 4 we have $q(f, r) \le cr^t$ when r is big enough, hence by Theorem 1 in [1] the derivative of $\frac{1}{f}$ takes every value b infinitely many times. □

Now we can derive the following Corollary 6.1 that gives the solution for a function of finite order, of the following general problem: given an entire function f, is it posible that all zeros of f' be zeros of f, but finitely many?

Corollary 6.1. *Let $f \in \mathcal{A}^0(\mathbb{K})$. Then f' admits infinitely many zeros that are not zeros of f.*

Theorem 7 is similar to a well known statement in complex analysis and its proof also is similar when $\rho(f) < +\infty$ [18] but is different when $\rho(f) = +\infty$.

Theorem 7. *Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$. Then $\rho(f) = \limsup_{n\to+\infty} \left(\frac{n\log(n)}{-\log|a_n|}\right)$.*

Remark. Of course, polynomials have a growth order equal to 0. On $\mathbb{K}$ as on $\mathbb{C}$ we can easily construct transcendental entire functions of order 0 or of order ∞.

Example 1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $-\log|a_n| \in [n(\log n)^2, n(\log n)^2 + 1]$. Then clearly,

$$\lim_{n\to+\infty} \frac{\log|a_n|}{n} = -\infty$$

hence the function $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence equal to $+\infty$. On the other hand,

$$\lim_{n\to+\infty} \frac{n\log n}{-\log|a_n|} = 0$$

hence $\rho(f) = 0$.

Example 2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $-\log|a_n| \in [n\sqrt{\log n}, n\sqrt{\log n}+1]$. Then

$$\lim_{n\to+\infty} \frac{\log|a_n|}{n} = -\infty$$

again and hence the function $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence equal to $+\infty$. On the other hand, $\lim_{n\to+\infty}\left(\frac{n\log n}{-\log|a_n|}\right) = +\infty$, hence $\rho(f) = +\infty$.

Here, we must recall a theorem proven in [3] and in [10] (Theorem 33.12), to characterize meromorphic functions admitting a primitive:

Theorem 8. *Let $f \in \mathcal{M}(\mathbb{K})$. Then f admits primitives if and only if all its residues are null.*

The following theorem was proven in 2011 with help of Jean-Paul Bezivin [1], [2]:

Theorem 9. *Let $f \in \mathcal{M}(\mathbb{K})$. Suppose that there exists $c,\ s \in]0,+\infty[$ such that $u(f,r) < cr^s\ \forall r > 1$. Then, for every $b \in \mathbb{K}$, $f' - b$ has infinitely many zeros.*

Thanks to Theorem 9, we can now prove Theorem 10:

Theorem 10. *Let $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$ with $g \in \mathcal{A}(\mathbb{K})$ and $h \in \mathcal{A}^0(\mathbb{K})$ and $\psi(h) < +\infty$. Then for every $b \in \mathbb{K}$, $f' - b$ has infinitely many zeros.*

Proof. Set $t = \rho(h)$. There exists $\ell > \psi(h)$ such that $q(r,h) \leq \ell r^t\ \forall r > 1$. Consequently, taking $s > t$ big enough, we have $u(f,r) < r^s\ \forall r > 1$ and hence f satisfies the hypotheses of Theorem 9. Therefore, for every $b \in \mathbb{K}$, $f' - b$ has infinitely may zeros. □

Corollary 10.1. *Let $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$ have all its residues null, with $g \in \mathcal{A}(\mathbb{K})$ and $h \in \mathcal{A}^0(\mathbb{K})$ and $\psi(h) < +\infty$. Then for every $b \in \mathbb{K}$, $f - b$ has infinitely many zeros.*

Remark. Consider a function f of the form $\sum_{n=1}^{\infty} \frac{1}{(x-a_n)^2}$ with $|a_n| = n^t$. Clearly f belongs to $\mathcal{M}(\mathbb{K})$, all residues are null, hence f admits primitives. Next, primitives satisfy the hypothesis of Theorem 9. Consequently, f takes every value infinitely many times. Therefore, f cannot be of the form $\frac{P}{h}$ with $P \in \mathbb{K}[x]$ and $h \in \mathcal{A}(\mathbb{K})$.

2. Type of growth

Definition and notation. In complex analysis, the type of growth is defined for an entire function of order $t < +\infty$ as $\sigma(f) = \limsup_{r\to+\infty} \frac{\log(M_f(r))}{r^t}$.

Of course the same notion may be defined for $f \in \mathcal{A}(\mathbb{K})$. Given $f \in \mathcal{A}^0(\mathbb{K})$ of order $t < +\infty$, we set $\sigma(f) = \limsup_{r\to+\infty} \frac{\log(|f|(r))}{r^t}$ and $\sigma(f)$ is called *the type of growth of* f.

Moreover, here we will also use the notation $\widetilde{\sigma}(f) = \liminf_{r\to+\infty} \frac{\log(|f|r))}{r^t}$.

Theorems 11, 12, 13 are proven in [4].

Theorem 11. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}^0(\mathbb{K})$ *such that* $\rho(f) \in]0, +\infty[$. *Then*

$$\sigma(f)\rho(f)e = \limsup_{n\to+\infty} \left(n \sqrt[n]{|a_n|^{\rho(f)}}\right).$$

Notation. Let $f \in \mathcal{A}(\mathbb{K})$, let $(a_n)_{n\in\mathbb{N}}$ be the sequence of zeros of f with $|a_n| \leq |a_{n+1}|$, $n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let w_n be the multiplicity order of a_n. For every $r > 0$, let $k(r)$ be the integer such that $|a_n| \leq r\ \forall n \leq k(r)$ and $|a_n| > r\ \forall n > k(r)$. . We set $\psi(f, r) = \frac{q(f,r)}{r^t}$ and $\sigma(f, r) = \sum_{n=0}^{k(r)} \frac{w_n(\log(r)-\log(c_n))}{r^t}$.

In the proof of Theorem 13, we will use the following trivial lemma:

Lemma L. *Let* g, h *be the real functions defined in* $]0, +\infty[$ *as* $g(x) = \frac{e^{tx}-1}{x}$ *and* $h(x) = \frac{1-e^{-tx}}{x}$ *with* $t > 0$. *Then:*

i) $\inf\{|g(x)| \ |x > 0\} = t$.
ii) $\sup\{|h(x)| \ |x > 0\} = t$.

Theorem 12. *Let* $f \in \mathcal{A}^0(\mathbb{K})$. *Then*

$$\rho(f) = \inf\{s \in]0, +\infty[\ |\ \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^s} = 0\}.$$

Proof. Indeed, let $M = \inf\{s \in]0, +\infty[\ |\ \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^s} = 0\}$. First we will prove that $\rho(f) \leq M$.

Let s be such that $\lim_{r\to+\infty} \frac{\log(|f|(r))}{r^s} = 0$. Let us fix $\epsilon > 0$. For r big enough, we have $\frac{\log(|f|(r))}{r^s} \leq \epsilon$, hence $\log(|f|(r)) \leq \epsilon r^s$, therefore $\log(\log(|f|(r))) \leq \log \epsilon + s\log(r)$, hence $\frac{\log(\log(|f|(r)))}{\log(r)} \leq s + \frac{\epsilon}{\log(r)}$. This is true for every $\epsilon > 0$, therefore $\limsup_{r\to+\infty} \frac{\log(\log(|f|(r)))}{\log(r)} \leq s$ i.e. $\rho(f) \leq s$ and hence, $\rho(f) \leq M$.

On the other hand, we notice that

$$M = \sup\left\{s \in]0, +\infty\left[\ \middle|\ \limsup_{r\to+\infty} \frac{\log(|f|(r))}{r^s} > 0\right.\right\}.$$

Now, suppose that for some $s > 0$, we have $\limsup_{r\to+\infty} \frac{\log(|f|(r))}{r^s} = b > 0$. Let us fix $\epsilon \in]0, b[$. There exists a sequence $(r_n)_{n\in\mathbb{N}}$ such that, when n is big enough, we have $b - \epsilon \leq \frac{\log(|f|(r_n))}{(r_n)^s} \leq b + \epsilon$, hence $s\log(r_n) + \log(b-\epsilon)$ $< \log(\log(|f|(r_n))) < s\log(r_n) + \log(b-\epsilon)$ therefore

$$s + \frac{\log(b-\epsilon)}{\log(r_n)} < \frac{\log(\log(|f|(r_n)}{\log(r_n)} < s + \frac{\log(b+\epsilon)}{\log(r_n)}.$$

Consequently, $\lim_{n\to+\infty} \frac{\log(\log(|f|(r_n))}{\log(r_n)} = s$ and therefore $\rho(f) \geq s$, hence $\rho(f) \geq M$. Finally, $\rho(f) = M$. □

Theorem 13. *Let $f,\ g \in \mathcal{A}^0(\mathbb{K})$. Then $\sigma(fg) \leq \sigma(f) + \sigma(g)$. If $\rho(f) \geq \rho(g)$, then $\sigma(f) \leq \sigma(fg)$. If $\rho(f) = \rho(g)$, then $\max(\sigma(f), \sigma(g)) \leq \sigma(fg)$.*

If $\rho(f) = \rho(g)$ and $\sigma(f) > \sigma(g)$ then $\rho(f+g) = \rho(f)$ and $\sigma(f+g) = \sigma(f)$. If $\rho(f+g) = \rho(f) \geq \rho(g)$ then $\sigma(f+g) \leq \max(\sigma(f), \sigma(g))$.

Proof. Let $s = \rho(f)$, $t = \rho(g)$ and suppose $s \geq t$. When r is big enough, we have $\max(\log(|f|(r)), \log(|g|(r)) \leq \log(|f.g|(r)) = \log(|f|(r)) + \log(|g|(r))$ and by Theorem 1, we have $\rho(fg) = s$. Therefore

$$\begin{aligned}\sigma(fg) &= \limsup_{r\to+\infty}\left(\frac{\log(|f.g|(r))}{r^s}\right)\\ &\leq \limsup_{r\to+\infty}\left(\frac{\log(|f|(r))}{r^s}\right) + \limsup_{r\to+\infty}\left(\frac{\log(|g|(r))}{r^s}\right)\\ &\leq \limsup_{r\to+\infty}\left(\frac{\log(|f|(r))}{r^s}\right) + \limsup_{r\to+\infty}\left(\frac{\log(|g|(r))}{r^t}\right) = \sigma(f) + \sigma(g).\end{aligned}$$

On the other hand,

$$\sigma(f) = \limsup_{r\to+\infty}\frac{\log(|f|(r))}{r^s} \leq \limsup_{r\to+\infty}\frac{\log(|fg|(r))}{r^s}.$$

But $\rho(fg) = s$, hence $\sigma(f) \leq \sigma(fg)$. Particularly, if $\rho(f) = \rho(g)$, then $\max(\sigma(f), \sigma(g)) \leq \sigma(fg)$.

Now, suppose $s > t$. Then by Corollary 1.1, $\rho(f+g) = \rho(f) = s$. Consequently,

$$\sigma(f+g) = \limsup_{r\to+\infty}\left(\frac{\log|f+g|(r)}{r^s}\right) \leq \limsup_{r\to+\infty}\left(\frac{\max(\log|f|(r), \log|g|(r))}{r^s}\right)$$

$$= \max\left(\limsup_{r\to+\infty}\left(\frac{\log|f|(r)}{r^s}\right), \limsup_{r\to+\infty}\left(\frac{\log|g|(r)}{r^s}\right)\right)$$

$$\leq \max\left(\limsup_{r\to+\infty}\left(\frac{\log|f|(r)}{r^s}\right), \limsup_{r\to+\infty}\left(\frac{\log|g|(r)}{r^t}\right)\right) = \max\big(\sigma(f), \sigma(g)\big).$$

Now, suppose $\rho(f) = \rho(g) = s$. Then

$$\max\left(\limsup_{r\to+\infty}\left(\frac{\log(|f|(r))}{r^s}\right), \limsup_{r\to+\infty}\left(\frac{\log(|g|(r))}{r^s}\right)\right) \le \limsup_{r\to+\infty}\left(\frac{\log(|f.g|(r))}{r^s}\right)$$

because the two both $|f|(r)$ and $|g|(r)$ tend to $+\infty$ with r. Consequently, $\sigma(fg) \ge \max(\sigma(f), \sigma(g))$.

Now, suppose again that $\rho(f) = \rho(g)$ and suppose $\sigma(f) > \sigma(g)$. Let $s = \rho(f)$, $b = \sigma(f)$. Then $b > 0$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to+\infty} r_n = +\infty$ and $\lim_{n\to+\infty}\frac{\log(|f|(r_n))}{(r_n)^s} = b$. Since $\sigma(g) < \sigma(f)$, we notice that when n is big enough we have $|g|(r_n) < |f|(r_n)$. Consequently, when n is big enough, we have $|f+g|(r_n) = |f|(r_n)$ and hence

$$\lim_{n\to+\infty}\frac{\log(|f+g|(r_n))}{(r_n)^s} = b. \tag{1}$$

By definition of σ we have $\sigma(f+g) \ge \lim_{n\to+\infty}\frac{\log(|f+g|(r_n))}{(r_n)^{\rho(f+g)}}$. By Theorem 1, we have $\rho(f+g) \le s$, hence

$$\begin{aligned}\sigma(f+g) &\ge \lim_{n\to+\infty}\frac{\log(|f+g|(r_n))}{(r_n)^{\rho(f+g)}} \ge \lim_{n\to+\infty}\frac{\log(|f+g|(r_n))}{(r_n)^s}\\ &= \lim_{n\to+\infty}\frac{\log(|f|(r_n))}{(r_n)^s} = \sigma(f)\end{aligned}$$

therefore by (1), $\sigma(f+g) \ge \sigma(f)$. Suppose that $\sigma(f+g) > \sigma(f)$. Putting $h = f+g$, we have $f = h-g$ with $\sigma(g) < \sigma(h)$, hence $\sigma(h-g) \ge \sigma(h)$ i.e. $\sigma(f) > \sigma(f+g)$, a contradiction. Consequently, $\sigma(f+g) = \sigma(f)$. Now we have, $\limsup_{r\to+\infty}\frac{\log(|f+g|(r))}{r^s} = b > 0$. But then, $\limsup_{r\to+\infty}\frac{\log(|f+g|(r))}{r^m} = 0$ $\forall m > s$. Therefore, by Therorem 11, $\rho(f+g) = \rho(f)$.

Finally, suppose now that $\rho(f+g) = \rho(f) \ge \rho(g)$. Let $s = \rho(f)$ and $t = \rho(g)$. Then,

$$\begin{aligned}\sigma(f+g) &= \limsup_{r\to+\infty}\frac{\log(|f+g|(r))}{r^s}\\ &\le \max\left(\limsup_{r\to+\infty}\frac{\log(|f|(r))}{r^s}, \limsup_{r\to+\infty}\frac{\log(|g|(r))}{r^s}\right) = \max(\sigma(f), \sigma(g)).\end{aligned}$$

□

Corollary 13.1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho(f) \ne \rho(g)$. Then $\sigma(f+g) \le \max(\sigma(f), \sigma(g))$.*

Proof. Indeed, suppose for instance $\rho(f) > \rho(g)$. Then $\rho(f+g) = \rho(f)$ hence, by the last assertion of Theorem 13, $\sigma(f+g) \le \max(\sigma(f), \sigma(g))$. □

Theorem 14. *Let $f \in \mathcal{A}^0(\mathbb{K})$ be not identically zero. Then*

$$\widetilde{\psi}(f) \leq \rho(f)\widetilde{\sigma}(f) \leq \rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)\Big(e\sigma(f) - \widetilde{\sigma}(f)\Big).$$

Moreover, if $\psi(f) = \lim_{r\to+\infty} \frac{q(f,r)}{r^{\rho(f)}}$ *or if* $\sigma(f) = \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^{\rho(f)}}$*, then* $\psi(f) = \rho(f)\sigma(f)$.

Proof. Without loss of generality we can assume that $f(0) \neq 0$. Let $t = \rho(f)$ and set $\ell = \log(|f(0|)$. Let $(a_n)_{n\in \mathbb{N}}$ be the sequence of zeros of f with $|a_n| \leq |a_{n+1}|$, $n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let w_n be the multiplicity order of a_n. For every $r > 0$, let $k(r)$ be the integer such that $|a_n| \leq r\ \forall n \leq k(r)$ and $|a_n| > r\ \forall n > k(r)$. Then by Theorem A, we have $\log(|f|(r)) = \ell + \sum_{n=0}^{k(r)} w_n(\log(r) - \log(|a_n|))$ hence

$$\sigma(f) = \limsup_{r\to+\infty} \Big(\frac{\ell + \sum_{n=0}^{k(r)} w_n(\log(r) - \log(|a_n|))}{r^t}\Big).$$

Given $r > 0$, set $c_n = |a_n|$ and let us keep the notations above. Then

$$\sigma(f) = \limsup_{r\to+\infty} \sigma(f,r), \quad \psi(f) = \limsup_{r\to+\infty} \psi(f,r). \tag{1}$$

We will first show the inequality $\rho(f)\sigma(f) \leq \psi(f)$. By the definition of $\sigma(f,r)$ we can derive

$$\sigma(f,r) \leq \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(\log(r) - \log(re^{-\alpha})\big)}{r^t} + \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(\log(re^{-\alpha}) - \log(c_n)\big)}{r^t} + \alpha \sum_{k(re^{-\alpha})<n\leq k(r)} \frac{w_n}{r^t}$$

therefore

$$\sigma(f,r) \leq \alpha \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{r^t} + \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n\big(\log(re^{-\alpha}) - \log(c_n)\big)}{r^t} + \alpha \sum_{k(re^{-\alpha})<n\leq k(r)} \frac{w_n}{r^t}$$

hence

$$\sigma(f,r) \leq \alpha \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{r^t} + e^{-t\alpha} \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(\log(re^{-\alpha}) - \log(c_n))}{(re^{-\alpha})^t} + \alpha \sum_{0\leq n\leq k(r)} \frac{w_n}{r^t} - \alpha \sum_{0\leq n\leq k(re^{-\alpha})} \frac{w_n}{r^t},$$

hence

$$\sigma(f,r) \le e^{-t\alpha} \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(\log(re^{-\alpha}) - \log(c_n))}{(re^{-\alpha})^t} + \alpha \sum_{0\le n\le k(r)} \frac{w_n}{r^t}.$$

Thus we have

$$\sigma(f,r) \le e^{-t\alpha}\sigma(f, re^{-\alpha}) + \alpha\psi(f,r).$$

We check that we can pass to superior limits on both sides, so we obtain $\sigma(f) \le e^{-t\alpha}\sigma(f) + \alpha\psi(f)$ therefore $\sigma(f)\frac{(1-e^{-t\alpha})}{\alpha} \le \psi(f)$. That holds for every $\alpha > 0$, hence

$$\psi(f) \ge \rho(f)\sigma(f). \tag{2}$$

Now, let us take again the definition of $\sigma(f,r)$. We have

$$\sigma(f,r) \ge \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(\log(r) - \log(re^{-\alpha}))}{r^t} + \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(\log(re^{-\alpha}) - \log(c_n))}{r^t}$$

hence

$$\sigma(f,r) \ge e^{-t\alpha} \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(\log(re^{-\alpha}) - \log(c_n))}{(re^{-\alpha})^t} + \alpha \sum_{0\le n\le k(r)} \frac{w_n}{r^t}$$

therefore

$$\sigma(f,r) \ge e^{-t\alpha}\sigma(f, r^{-\alpha}) + \alpha e^{-t\alpha}\psi(r, e^{-\alpha})$$

and hence, passing to the inferior limit on both sides we obtain

$$\widetilde{\psi}(f) \le \frac{1 - e^{-t\alpha}}{\alpha e^{)t\alpha}}\widetilde{\sigma}(f)$$

and therefore by Lemma L ii) we have $\widetilde{\psi}(f) \le \rho(f)\widetilde{\sigma}(f)$.

We will now show the inequality

$$\psi(f) \le \rho(f)(e\sigma(f) - \widetilde{\sigma}(f)).$$

Let us fix $\alpha > 0$. We can write

$$\sigma(f,r) = \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n(\log(r) - \log(re^{-\alpha}))}{r^t}$$
$$+ \sum_{j=0}^{k(re^{-\alpha})} \frac{w_j(\log(re^{-\alpha}) - \log(c_n))}{r^t} + \sum_{k(re^{-\alpha})<j\le k(r)} \frac{w_j(\log(r) - \log(c_j))}{r^t}$$

hence

$$\sigma(f,r) \geq \alpha \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{r^t} + \sum_{j=0}^{k(re^{-\alpha})} \frac{w_j(\log(re^{-\alpha}) - \log(c_n))}{r^t}$$

hence

$$\sigma(f,r) \geq \alpha e^{-t\alpha} \sum_{n=0}^{k(re^{-\alpha})} \frac{w_n}{(re^{-t\alpha})} + e^{-t\alpha} \sum_{j=0}^{k(re^{-\alpha})} \frac{w_n(\log(re^{-\alpha}) - \log(c_n))}{(re^{-\alpha})^t}$$

and hence

$$\sigma(f,r) \geq \alpha\psi(f, re^{-\alpha}) + e^{-t\alpha}\sigma(f, re^{-\alpha}).$$

Therefore, we can derive

$$\alpha e^{-t\alpha}\psi(f) \leq \limsup_{r\to+\infty} \Big(\sigma(f,r) - e^{-t\alpha}\sigma(f, re^{-\alpha}))\Big)$$

and therefore

$$\alpha e^{-t\alpha}\psi(f) \leq \sigma(f) - e^{-t\alpha}\widetilde{\sigma}(f)).$$

That holds for every $\alpha > 0$ and hence, when $t\alpha = 1$, we obtain $\psi(f) \leq \rho(f)\big(e\sigma(f) - \widetilde{\sigma}(f)\big)$ which is the left hand inequality of the general conclusion.

Now, suppose that $\sigma(f) = \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^t}$. Then we have $\limsup_{r\to+\infty} \psi(f,r) \leq \sigma(f)\Big(\frac{e^{t\alpha}-1}{\alpha}\Big)$ and hence $\psi(f) \leq \sigma(f)\Big(\frac{e^{t\alpha}-1}{\alpha}\Big)$. That holds for every $\alpha > 0$ and then we obtain $\psi(f) \leq t\sigma(f)$, i.e. $\psi(f) \leq \rho(f)\sigma(f)$, hence by (2), $\psi(f) = \rho(f)\sigma(f)$.

Now, suppose that

$$\psi(f) = \lim_{r\to+\infty} \sum_{n=0}^{k(r)} \frac{w_n}{r^t} = \lim_{r\to+\infty} \psi(f,r).$$

We can obviously find a sequence $(r_n)_{n\in\mathbb{N}}$ in $]0,+\infty[$ of limit $+\infty$ such that $\sigma(f) = \lim_{n\to+\infty} \sigma(f, r_n e^{-\alpha})$. Then, by (1) we have

$$\sigma(f, r_n) \geq \alpha e^{-t\alpha}\psi(f, \frac{r_n}{e^\alpha}) + e^{-t\alpha}\sigma(f, \frac{r_n}{e^\alpha})$$

hence

$$\limsup_{n\to+\infty} \sigma(f, r_n) \geq \alpha e^{-t\alpha}\psi(f) + e^{-t\alpha}\sigma(f)$$

and hence

$$\sigma(f) \geq \alpha e^{-t\alpha}\psi(f) + e^{-t\alpha}\sigma(f)$$

therefore $\psi(f) \leq \Big(\frac{e^{t\alpha}-1}{\alpha}\Big)\sigma(f)$. Finally, we have, $\psi(f) \leq \rho(f)\sigma(f)$ and hence by (2), $\psi(f) = \rho(f)\sigma(f)$. □

Remark. 1) When neither σ nor ψ are obtained as veritable limits when r tends to $+\infty$, the method does not let us prove that $\psi = \rho\sigma$, the natural conjecture.

2) Concerning the upper bound $\psi(f) \le \rho(f)(e\sigma(f) - \widetilde{\sigma}(f))$ it is possible to improve this a bit by defining the number $u_0 > 0$ such that $e^{u_0}(u_0 - 1) = -\frac{\widetilde{\sigma}(f)}{\sigma(f)}$ and then we have

$$\psi(f) \le \frac{\rho(f)(e^{u_0}\sigma(f) - \widetilde{\sigma}(f))}{u_0}.$$

Corollary 14.1. *Let $f \in \mathcal{A}(\mathbb{K})$ be not identically zero and have finite growth order. Then $\sigma(f)$ is finite if and only if so is $\psi(f)$.*

Corollary 14.2 *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}^0(\mathbb{K})$ be not identically zero and have finite growth order. If $\widetilde{\psi}(f) = \psi(f)$, then $\widetilde{\sigma}(f) = \sigma(f)$.*

By Theorem 11 we can also notice this corollary:

Corollary 14.3 *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}^0(\mathbb{K})$ be not identically zero. If $\psi(f) = \lim_{r\to+\infty} \frac{q(f,r)}{r^{\rho(f)}}$ or if $\sigma(f) = \lim_{r\to+\infty} \frac{\log(|f|(r))}{r^{\rho(f)}}$, then*

$$e\psi(f) = \limsup_{n\to+\infty} \left(n \sqrt[n]{|a_n|^{\rho(f)}}\right).$$

We will now present Example 3 where neither $\psi(f)$ nor $\sigma(f)$ are obtained as limits but only as superior limits: we will show that the equality $\psi(f) = \rho(f)\sigma(f)$ holds again.

Example 3. Let $r_n = 2^n$, $n \in \mathbb{N}$ and let $f \in \mathcal{A}(\mathbb{K})$ have exactly 2^n zeros in $C(0, r_n)$ and satisfy $f(0) = 1$. Then $q(f, r_n) = 2^{n+1} - 1\ \forall n \in \mathbb{N}$. We can see that the function $h(r)$ defined in $[r_n, r_{n+1}[$ by $h(r) = \frac{q(f,r)}{r}$ is decreasing and satisfies $h(r_n) = \frac{2^{n+1}-2}{2^n}$ and $\lim_{r\to r_{n+1}} \frac{h(r)}{r} = \frac{2^{n+1}-2}{2^{n+1}}$. Consequently, $\limsup_{r\to+\infty} h(r) = 2$ and $\liminf_{r\to+\infty} h(r) = 1$. Particularly, by Theorem 4, we have $\rho(f) = 1$ and of course $\psi(f) = 2$. On the other hand, we can show that $\sigma(f) = 2$.

Now, Theorem 14 and Example 3 suggest the following conjecture:

Conjecture 1. *Let $f \in \mathcal{A}^0(\mathbb{K})$ be such that either $\sigma(f) < +\infty$ or $\psi(f) < +\infty$. Then $\psi(f) = \rho(f)\sigma(f)$.*

Notation. Henceforth, we will denote by $\beta(t)$ the solution of $\mathcal{E}$.

Now, by Corollary 9.1, we can also state Corollary 14.4:

Corollary 14.4. *Let* $f = \dfrac{g}{h} \in \mathcal{M}(\mathbb{K})$*, with* $g,\ h \in \mathcal{A}(\mathbb{K})$ *not identically zero and be such that* h *has finite order of growth and and finite type of growth. Then* f' *takes every value* $b \in \mathbb{K}$ *infinitely many times.*

3. Order and type of the derivative

Theorems 15, 16, 17, 18 are proven in [4].

Theorem 15. *Let* $f \in \mathcal{A}(\mathbb{K})$ *be not identically zero. Then* $\rho(f) = \rho(f')$*.*

Corollary 15.1 *The derivation on* $\mathcal{A}(\mathbb{K})$ *restricted to the algebra* $\mathcal{A}(\mathbb{K}, t)$ *(resp. to* $\mathcal{A}^0(\mathbb{K})$*) provides that algebra with a derivation.*

In complex analysis, it is known that if an entire function f has order $t < +\infty$, then f and f' have same type. We will check that it is the same here.

Theorem 16. *Let* $f \in \mathcal{A}(\mathbb{K})$ *be not identically zero, of order* $t \in]0, +\infty[$*. Then* $\sigma(f) = \sigma(f')$*.*

By Theorems 14, 15, 16 we can now derive

Corollary 16.1. *Let* $f \in \mathcal{A}^0(\mathbb{K})$ *be not identically zero, of order* $t < +\infty$*. Then*
$\rho(f)\sigma(f) \le \psi(f') \le e\rho(f)\sigma(f)$,
$|\psi(f') - \psi(f)|_\infty \le (e-1)\rho(f)\sigma(f)$,
$\frac{1}{e-1} \le \frac{\psi(f')}{\psi(f)} \le e - 1$.

Corollary 16.2. *Let* $f \in \mathcal{A}^0(\mathbb{K})$ *be not identically zero, of order* $t < +\infty$*. Then* $\rho(f)\sigma(f) \le \psi(f') \le e\rho(f)\sigma(f)$*. Moreover, in each one of the following hypotheses, we have* $\psi(f') = \psi(f) = \rho(f)\sigma(f)$*:*

i) $\psi(f) = \lim\limits_{r\to+\infty} \psi(f, r)$ *and* $\psi(f') = \lim\limits_{r\to+\infty} \psi(f', r)$,
ii) $\sigma(f) = \lim\limits_{r\to+\infty} \sigma(f, r)$ *and* $\sigma(f') = \lim\limits_{r\to+\infty} \sigma(f', r)$,
iii) $\psi(f) = \lim\limits_{r\to+\infty} \psi(f, r)$ *and* $\sigma(f') = \lim\limits_{r\to+\infty} \sigma(f', r)$,
iv) $\sigma(f) = \lim\limits_{r\to+\infty} \sigma(f, r)$ *and* $\psi(f') = \lim\limits_{r\to+\infty} \psi(f', r)$.

Corollary 16.3. *Let* $f = \frac{g}{h} \in \mathcal{M}(\mathbb{K})$ *be not identically zero, with* $g,\ h \in \mathcal{A}(\mathbb{K})$*, having all residues null and such that* h *has finite order of growth and finite type of growth. Then* f *takes every value* $b \in \mathbb{K}$ *infinitely many times.*

Conjecture 1 suggests and implies the following Conjecture 2:

Conjecture 2. $\psi(f) = \psi(f')\ \forall f \in \mathcal{A}^0(\mathbb{K})$.

Theorem 17. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be transcendental and of same order $t \in [0, +\infty[$. Then for every $\epsilon > 0$, we have*

$$\limsup_{r\to+\infty}\Big(\frac{r^{\epsilon}q(g,r)}{q(f,r)}\Big) = +\infty.$$

Remark. Comparing the number of zeros of f' to this of f inside a disk is very uneasy. Now, we can give some precisions. By Theorem 17 we can derive Corollary 17.1.

Corollary 17.1. *Let $f \in \mathcal{A}^0(\mathbb{K})$ be not affine. Then for every $\epsilon > 0$, we have*

$$\limsup_{r\to+\infty}\Big(\frac{r^{\epsilon}q(f',r)}{q(f,r)}\Big) = +\infty$$

and

$$\limsup_{r\to+\infty}\Big(\frac{r^{\epsilon}q(f,r)}{q(f',r)}\Big) = +\infty.$$

Corollary 17.2. *Let $f \in \mathcal{A}^0(\mathbb{K})$. Then $\psi(f)$ is finite if and only if so is $\psi(f')$.*

We can now give a partial solution to a problem that arose when studying the zeros of derivatives of meromorphic functions: given $f \in \mathcal{A}(\mathbb{K})$, is it possible that f' divides f in the algebra $\mathcal{A}(\mathbb{K})$? Theorem 18 is proven in [4].

Theorem 18. *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$. Suppose that for some number $s > 0$ we have $\limsup_{r\to+\infty} |q(f,r)|r^s > 0$ (where $|q(f,r)|$ is the absolute value of $q(f,r)$ defined on $\mathbb{K}$). Then f' has infinitely many zeros that are not zeros of f.*

Remark. It is possible to deduce the proof of Theorem 18 by using Lemma 1.4 in [4].

Corollary 18.1. *Let $f \in A^0(\mathbb{K}) \setminus \mathbb{K}[x]$. Then f' has infinitely many zeros that are not zeros of f.*

Corollary 18.2. *Let $f \in \mathcal{A}^0(\mathbb{K}) \setminus \mathbb{K}[x]$. Then f' does not divide f in $\mathcal{A}(\mathbb{K})$.*

Corollary 18.3 is a very partial answer to the p-adic Hayman conjecture when $n = 1$, which is not solved yet [12], [17].

Corollary 18.3. *Let $f \in \mathcal{M}(\mathbb{K})$ be such that*

$$\limsup_{r\to+\infty} |q(\frac{1}{f}, r)|r^s > 0$$

for some $s > 0$. Then ff' has at least one zero.

Proof. Indeed, suppose that ff' has no zero. Then f is of the form $\frac{1}{h}$ with $h \in \mathcal{A}(\mathbb{K})$ and $f' = -\frac{h'}{h^2}$ has no zero, hence every zero of h' is a zero of h, a contradiction to Theorem 17 since $\limsup_{r\to+\infty} |q(h,r)|r^s > 0$. □

Remark. Concerning complex entire functions, we check that the exponential is of order 1 but is divided by its derivative in the algebra of complex entire functions.

It is also possible to derive Corollary 17.3 from Theorem 1 in the paper by Jean-Paul Bezivin, Kamal Boussaf and me. Indeed, let $g = \frac{1}{f}$. By Theorem 4, $\limsup_{r\to+\infty} \frac{q(f,r)}{r^t}$ is a finite number. Consequently, there exists $c > 0$ such that $q(f,r) \le cr^t$ $\forall r > 1$ and therefore the number of poles of g in $d(0,r)$ is upper bounded by cr^t whenever $r > 1$. Consequently, we can apply Theorem 8 and hence the meromorphic function g' has infinitely many zeros. Now, suppose that f' divides f in $\mathcal{A}(\mathbb{K})$. Then every zero of f' is a zero of f with an order superior, hence $\frac{f'}{f^2}$ has no zero, a contradiction.

If the residue characteristic of $\mathbb{K}$ is $p \neq 0$, we can easily construct an example of entire function f of infinite order such that f' does not divide f in $\mathcal{A}(\mathbb{K})$. Let $f(x) = \prod_{n=0}^{\infty} \left(1 - \frac{x}{\alpha_n}\right)^{p^n}$ with $|\alpha_n| = n+1$. We check that $q(f, n+1) = \sum_{k=0}^{n} p^k$ is prime to p for every $n \in \mathbb{N}$. Consequently, Theorem 16 shows that f is not divided by f' in $\mathcal{A}(\mathbb{K})$. On the other hand, fixing $t > 0$, we have

$$\frac{q(f, n+1)}{(n+1)^t} \ge \frac{p^n}{(n+1)^t}$$

hence

$$\limsup_{r\to+\infty} \frac{q(f,r)}{r^t} = +\infty \ \forall t > 0$$

therefore, f is not of finite order.

Theorem 18 suggests the following conjecture:

Conjecture 3. *Given $f \in \mathcal{A}(\mathbb{K})$ (other than $(x-a)^m$, $a \in \mathbb{K}$, $m \in \mathbb{N}$) there exists no $h \in \mathcal{A}(\mathbb{K})$ such that $f = f'h$.*

4. Application to branched values

Let us recall some definitions concerning perfectly branched values. They were rst introduced in complex analysis [7]. Here we can give the same definition in p-adic analysis.

Definition. Let $f \in \mathcal{M}(IK) \setminus IK(x)$. A value $b \in IK$ is called *a perfectly branched value for* f if all zeros of $f?b$ are multiple except nitely many and b is called *a totally branched value for* f if all zeros of $f?b$ are multiple, without any exception. Let us recall that a complex meromorphic function f admits at most 4 perfectly branched values. And a complex entire function f admits at most 2 perfectly branched values. In complex analysis, these results are sharp: indeed, the Weierstrass unction admits 4 pefectly branched values (taking the innite as a value) and the functions $\sin, \cos$ admit two perfectly branched values: 1 and ?1.

Concerning meromorphic function on IK, we have this theorem [11]:

Theorem 19. *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. Then f admits at most 4 perfectly branched values and at most 3 totally branched values. Moreover, if f has finitely many poles, then f admits at most 1 perfectly branched value. Particularly, if $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$, then f admits at most one perfectly branched value.*

Remark. We dont know any meromorphic function on IK admitting 4 perfectly branched values. In [12], a meromorphic function on IK admitting 3 totally branched values is constructed.

We will now obtain new results with the use of the p-adic Nevanlinna Theory [6]. Here we only need to recall the denition of the Nevanlinna characteristic function of an entire function.

Definition. Let $f \in \mathcal{A}(\mathbb{K})$ having no zero at 0, let $(an)_{n\in \mathbb{N}}$ be the sequence of zeros of f such that $|a_n||a_{n+1}| \forall n \in \mathbb{N}$ and let s_nn be the multiplicity order of a_n . For each $r > 0$, we put $T(r, f) = \sum_{|a_n|r} q_n(log(r) - log(a_n))$.

Theorem 20. [5] *Let $f,\ g \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ be such that* $\limsup_{r\to+\infty} \frac{T(r,f)}{T(r,g)} > 2$. *Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched values.*

Corollary 20.1. *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$, (resp. let $f \in \mathcal{A}_u(d(0, R^-))$) and let $g \in \mathcal{A}_f(\mathbb{K})$, (resp. $g \in \mathcal{A}_f(d(0, R^-))$). Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched values.*

Theorems 1 suggests the following conjecture that we cannot prove due to the absence of a p-adic Yamanoi's theorem:

Conjecture 4. *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. There exists at most four small functions $w \in \mathcal{M}_f(\mathbb{K})$ that are perfectly branched with respect to f. Moreover, if $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ then there exists at most two small functions $w \in \mathcal{A}_f(\mathbb{K})$ that are perfectly branched with respect to f.*

The next theorems use the growth order and the growth type for p-adic entire functions. Indeed, in order to obtain some results on branched small functions for p-adic meromorphic functions, since we don't enjoy a Yamanoi-Nevanlinna theorem, we will use another strategy combining the order of growth and the type of growth for entire functions, thanks to the link between the type of growth and the Nevanlinna characteristic function, for an entire function.

Here, we will use the compared growth of numerator and denominator of a p-adic meromorphic function in order to examine how many perfectly branched values it can admit.

Theorem 21. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho(f) > \rho(g)$. Then*

$$\liminf_{r \to +\infty} \frac{T(r,g)}{T(r,f)} = 0.$$

By Theorem 20, we can now derive Corollary 21.1.

Corollary 21.1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho(f) \neq \rho(g)$. Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two distinct perfectly branched values.*

Now, when $\rho(f) = \rho(g)$, we can still give some precision.

Theorem 22. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ and suppose that $\rho(f) = \rho(g) \in]0, +\infty[$ and $\sigma(f) \neq \sigma(g)$. Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most three distinct perfectly branched values. Moreover, if $2\sigma(g) < \sigma(f)$ or if $2\sigma(f) < \sigma(g)$ then $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched values.*

Corollary 22.1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\frac{f}{g}$ admits four distinct branched values. Then $\rho(f) = \rho(g)$. Moreover, if $\rho(f) \in]0, +\infty[$, then $\sigma(f) = \sigma(g)$.*

References

[1] J. P. Bezivin, K. Boussaf and A. Escassut, *Zeros of the derivative of a p-adic meromorphic function,* Bulletin des Sciences Mathématiques, 136 (8) (2012), 839–847.

[2] J. P. Bezivin, K. Boussaf and A. Escassut, *Some new and old results on zeros of the derivative of a p-adic meromorphic function,* Contemp. Math., Amer. Math. Soc., 596 (2013), 23–30.

[3] K. Boussaf, A. Escassut and J. Ojeda, *Primitives of p-adic meromorphic functions*, Contemp. Math., 551 (2011), 51–56

[4] K. Boussaf, A. Boutabaa and A. Escassut, *Growth of p-adic entire functions and applications*, Houston J. Math., 40 (3) (2014), 715–736.
[5] K. Boussaf, A. Escassut and J. Ojeda. *Growth of complex and p-adic meromorphic func- tions and branched small functions*, Bull. Belg. Math. Soc. - Simon Stevin, (2016).
[6] A. Boutabaa, *Theorie de Nevanlinna p-adique*, Manuscripta Math., 67 (1990), 251–269.
[7] K. S. Charak, *Value distribution theory of meromorphic functions*, Math. Newsl., 18 (4) (2009), 1–35.
[8] A. Escassut, *Analytic Elements in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
[9] A. Escassut, *p-adic Value Distribution*. Some Topics on Value Distribution and Differentability in Complex and P-adic Analysis, p. 42- 138. Math. Monogr., Series 11. Science Press.(Beijing 2008).
[10] A. Escassut, *Value Distribution in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (2015).
[11] A. Escassut and J. Ojeda, *Exceptional values of p-adic analytic functions and derivative*, Complex Var. Elliptic Equ., 56 (1-4) (2011), 263–269.
[12] A. Escassut and J. Ojeda, *Branched values and quasi-exceptional values for p-adic mermorphic functions*, Houston J. Math., 39 (3) (2013), 781–795.
[13] A. Escassut and J. Ojeda, *The p-adic Hayman conjecture when $n = 2$*, Complex Var. Elliptic Equ., 59 (10) (2014), 1452–1455.
[14] P. C. Hu and C. C. Yang, *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers, (2000).
[15] M. Krasner, *Prolongement analytique uniforme et multiforme dans les corps valués complets: éléments analytiques, préliminaires du théeorème dunicité*. CRAS Paris A, 239 (1954), 468–470.
[16] M. Krasner *Prolongement analytique uniforme et multiforme dans les corps valués complets*, Les tendances géométriques en algèbre et théorie des nombres, Clermont-Ferrand.
[17] J. Ojeda, *On Hayman's Conjecture over a p-adic field*, Taiwanese J. Math., 12 (9) (2008), 2295–2313.
[18] Lee A. Rubel, *Entire and Meromorphic Functions*, Springer-Verlag, New York, (1996).

(Received: June 20, 2016)
(Revised: September 5, 2016)

Laboratoire de Mathématiques, UMR 6620
Unversité Blaise Pascal
24 Avenue des Landais
63178 Aubiere Cedex
France

kamal.boussaf@math.univ-bpclermont.fr
abdelbaki.boutabaa@math.univ-bpclermont.fr
alain.escassut@math.univ-bpclermont.fr