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EXISTENCE OF PERIODIC SOLUTIONS FOR TOTALLY NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY

ABDELOUAHEB ARDJOUNI AND AHCÉNE DJOUDI

Abstract. We use a modification of Krasnoselskii's fixed point theorem introduced by T. A. Burton (see [1] Theorem 3) to show that the totally nonlinear neutral differential equation with variable delay

$$x'(t) = -a(t)x^3(t) + c(t)x'(t-g(t))Q'\left(x\left(t-g(t)\right)\right) + G\left(t, x^3(t), x^3(t-g(t))\right),$$

has a periodic solution. We invert this equation to construct a sum of a compact map and a large contraction which is suitable for applying the modification of Krasnoselskii's theorem. The results of [5] are generalized. Finally, an example is given to illustrate our result.

1. Introduction

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of a differential equation. Motivated by the papers [1,3-8] and the references therein, we consider the following totally nonlinear neutral differential equation with variable delay

$$x'(t) = -a(t)x^3(t)+c(t)x'(t-g(t))Q'\left(x\left(t-g(t)\right)\right)+G\left(t, x^3(t), x^3(t-g(t))\right), \tag{1.1}$$

where a is real valued function, c is continuously differentiable, g is twice continuously differentiable, $Q : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous with respect to its arguments. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due T. A. Burton (see [1], Theorem 3) to show the existence of periodic solutions for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters cannot be applied directly. Then, we resort to the idea of adding and

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subtracting a linear term. As noted by Burton in [1], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, to show the existence of a periodic solution. For details on Krasnoselskii theorem we refer the reader to [9]. A special case of equation (1.1) with $Q(x) = x$ and where the delay g is some positive continuous and periodic function has been recently studied in [5]. In Section 2, we present the inversion of equation (1.1) and the modification of Kransnoselskii's fixed point theorem. We present our main results on periodicity in Section 3, and we provide an example to illustrate our result. The results presented in this paper generalize the main results in [5].

2. Inversion of equation (1.1)

Let $T > 0$ and define $C_T = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real valued functions. C_T is a Banach space endowed with the norm

$$\|\varphi\| = \max_{0\leq t\leq T} |\varphi(t)|.$$

In this paper assume that

$$a(t+T) = a(t),\ c(t+T) = c(t),\ G(t+T,x,y) = G(t,x,y),\ g(t+T) = g(t), \tag{2.1}$$

where a and G are continuous functions, c continuously differentiable, g twice continuously differentiable and $g(t) \geq g^* > 0$. Also, we assume

$$\int_0^T a(s)ds > 0. \tag{2.2}$$

Functions $Q(x)$, $Q'(x)$ and $G(t,x,y)$ are locally Lipschitz continuous in x, x and in x and y, respectively. That is, there are positive constants k_1, k_2, k_3, k_4 so that if $|x|, |y|, |z|, |w| \leq L$, then

$$|Q(x) - Q(y)| \leq k_1 \|x - y\|, \tag{2.3}$$

$$|Q'(x) - Q'(y)| \leq k_2 \|x - y\|, \tag{2.4}$$

and

$$|G(t,x,y) - G(t,z,w)| \leq k_3 \|x - z\| + k_4 \|y - w\|. \tag{2.5}$$

Also, we assume that for all t, $0 \leq t \leq T$,

$$g'(t) \neq 1. \tag{2.6}$$

Lemma 1. *Suppose* (2.1) , (2.2) *and* (2.6) *hold. If* $x \in C_T$, *then* x *is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) &= \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \int_{t-T}^{t} a(u)[x(u) - x^3(u)] e^{-\int_u^t a(s)ds} du \\ &+ \frac{c(t)}{(1 - g'(t))} Q(x(t - g(t))) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \\ &\times \int_{t-T}^{t} \left[G(u, x^3(u), x^3(u - g(u))) - h(u)Q(x(u - g(u)))\right] e^{-\int_u^t a(s)ds} du, \end{aligned} \tag{2.7}$$

where

$$h(u) = \frac{(c'(u) + a(u)c(u))(1 - g'(u)) + c(u)g''(u)}{(1 - g'(u))^2}. \tag{2.8}$$

Proof. Let x be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} x'(t) + a(t)x(t) &= a(t)x(t) - a(t)x^3(t) \\ &+ c(t)x'(t - g(t))Q'(x(t - g(t))) + G(t, x^3(t), x^3(t - g(t))). \end{aligned}$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate from $t - T$ to t to obtain

$$\begin{aligned} \int_{t-T}^{t} \left[x(u)e^{\int_0^u a(s)ds}\right]' du &= \int_{t-T}^{t} a(u)[x(u) - x^3(u)] e^{\int_0^u a(s)ds} du \\ &+ \int_{t-T}^{t} G(u, x^3(u), x^3(u - g(u))) e^{\int_0^u a(s)ds} du \\ &+ \int_{t-T}^{t} c(u)x'(u - g(u))Q'(x(u - g(u))) e^{\int_0^u a(s)ds} du. \end{aligned}$$

Rewrite the last term as

$$\begin{aligned} &\int_{t-T}^{t} c(u)x'(u - g(u))Q'(x(u - g(u))) e^{\int_0^u a(s)ds} du \\ &= \int_{t-T}^{t} \frac{c(u)(1 - g'(u))x'(u - g(u))Q'(x(u - g(u)))}{(1 - g'(u))} e^{\int_0^u a(s)ds} du. \end{aligned}$$

Using integration by parts, and that c, g and x are periodic we obtain

$$\int_{t-T}^{t} c(u)x'(u-g(u))Q'\left(x\left(u-g(u)\right)\right)e^{\int_0^u a(s)ds}du$$
$$= \frac{c\left(t\right)}{\left(1-g'\left(t\right)\right)}Q\left(x\left(t-g(t)\right)\right)e^{\int_0^t a(s)ds}\left(1-e^{-\int_{t-T}^t a(s)ds}\right)$$
$$-\int_{t-T}^{t} h(u)Q\left(x\left(u-g(u)\right)\right)e^{\int_0^u a(s)ds}du$$

where h is given by (2.8). We arrive at

$$x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} = \int_{t-T}^{t} a(u)[x(u)-x^3(u)]e^{\int_0^u a(s)ds}du$$
$$+\frac{c(t)}{(1-g'(t))}Q\left(x(t-g(t))\right)e^{\int_0^t a(s)ds}\left(1-e^{-\int_{t-T}^t a(s)ds}\right)$$
$$+\int_{t-T}^{t} G(u,x^3\left(u\right),x^3(u-g(u)))e^{\int_0^u a(s)ds}du$$
$$-\int_{t-T}^{t} h(u)Q(x(u-g(u)))e^{\int_0^u a(s)ds}du.$$

Now, the lemma follows by dividing both sides of the above equation by $e^{\int_0^t a(s)ds}$ and using the fact that $x(t)=x(t-T)$. □

Krasnoselskii (see [2] or [9]) combined the contraction mapping theorem and Shauder's theorem and formulated the following hybrid and attractive result.

Theorem 1. *Let M be a closed convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map M into S such that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,
(ii) *A is continuous and AM is contained in a compact set,*
(iii) *B is a contraction with constant $\alpha < 1$.*

Then there is a $z \in M$ with $z = Az + Bz$.

This is a captivating result and has a number of interesting applications. In recent years much attention has been paid to this theorem. T. A. Burton [2] observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated in Theorem 3 bellow (see [3] for the proof).

Definition 1. *Let (M, d) be a metric space and $B : M \to M$. B is said to be a large contraction if $\varphi, \psi \in M$, with $\varphi \neq \psi$ then $d(B\varphi, B\psi) < d(\varphi, \psi)$*

and if for all $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi, \psi \in M,\ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 2. *Let (M, d) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 3. *Let M be a bounded convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A, B map M into M and that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in M$ with $z = Az + Bz$.

We will use this theorem to prove the existence of periodic solutions for (1.1).

3. Existence of periodic solutions

To apply Theorem 3, we need to define a Banach space S, a bounded convex subset M of S and construct two mappings. One is a large contraction and the other is completely continuous. So, we let $(S, \|.\|) = (C_T, \|.\|)$ and $M = \{\varphi \in S : \|\varphi\| \leq L,\ \|\varphi'\| \leq L'\}$, where $L = \sqrt{3}/3$ and L' is a positive constant. So we have to express (2.7) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (H\varphi)(t),$$

where $A, B : S \to S$ are defined by

$$(B\varphi)(t) := \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \int_{t-T}^{t} a(u) \left[\varphi(u) - \varphi^3(u)\right] e^{-\int_u^t a(s)ds} du, \tag{3.1}$$

and

$$(A\varphi)(t) := \frac{c(t)}{(1 - g'(t))} Q\left(\varphi(t - g(t))\right) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}$$
$$\times \int_{t-T}^{t} \left[G(u, \varphi^3(u), \varphi^3(u - g(u))) - h(u) Q(\varphi(u - g(u)))\right] e^{-\int_u^t a(s)ds} du. \tag{3.2}$$

We need the following assumptions

$$\left[(k_3+k_4)\,L^3+|G(t,0,0)|\right]\leq \beta La(t), \tag{3.3}$$

$$|h(t)|\,(k_1L+|Q\,(0)|)\leq \delta La(t), \tag{3.4}$$

$$\max_{t\in[0,T]}\left|\frac{c(t)}{(1-g'(t))}\right|=\alpha, \tag{3.5}$$

$$J\left[\alpha\left(k_1+\frac{|Q\,(0)|}{L}\right)+\beta+\delta\right]\leq 1, \tag{3.6}$$

where α,β,δ and J are constants with $J\geq 3$.

We begin with the following Proposition (see [1], [2]).

Proposition 1. *Let* $\|.\|$ *be the supremum norm,*

$$M=\{\varphi\in S:\|\varphi\|\leq\sqrt{3}/3,\ \left\|\varphi'\right\|\leq L'\},$$

and define $(\mathfrak{B}\varphi)\,(t):=\varphi\,(t)-\varphi^3\,(t)$. *Then* $\mathfrak{B}$ *is a large contraction of the set* M.

Proof. For each $t\in\mathbb{R}$ we have, for φ,ψ real functions,

$$\begin{aligned}|(\mathfrak{B}\varphi)(t)-(\mathfrak{B}\psi)(t)|&=|\varphi(t)-\varphi^3(t)-\psi(t)+\psi^3(t)|\\&=|\varphi(t)-\psi(t)||1-(\varphi^2(t)+\varphi(t)\psi(t)+\psi^2(t))|.\end{aligned}$$

Then for

$$|\varphi(t)-\psi(t)|^2=\varphi^2(t)-2\varphi(t)\psi(t)+\psi^2(t)\leq 2(\varphi^2(t)+\psi^2(t))$$

and for $\varphi^2(t)+\psi^2(t)<1$, we have

$$\begin{aligned}|(\mathfrak{B}\varphi)(t)-(\mathfrak{B}\psi)(t)|&=|\varphi(t)-\psi(t)|\left[1-(\varphi^2(t)+\psi^2(t))+|\varphi(t)\psi(t)|\right]\\&\leq|\varphi(t)-\psi(t)|\left[1-(\varphi^2(t)+\psi^2(t))+\frac{\varphi^2(t)+\psi^2(t)}{2}\right]\\&\leq|\varphi(t)-\psi(t)|\left[1-\frac{\varphi^2(t)+\psi^2(t)}{2}\right].\end{aligned}$$

Thus, $\mathfrak{B}$ is pointwise a large contraction. But the application $\mathfrak{B}$ is still a large contraction for the supremum norm. Let $\varepsilon\in(0,1)$ and $\varphi,\psi\in M$ where $\|\varphi-\psi\|\geq\varepsilon$.

(a) Suppose that for some t we have $\varepsilon/2\leq|\varphi(t)-\psi(t)|$. Then

$$(\varepsilon/2)^2\leq|\varphi(t)-\psi(t)|^2\leq 2(\varphi^2(t)+\psi^2(t)),$$

that is

$$\varphi^2(t)+\psi^2(t)\geq\varepsilon^2/8.$$

For all such t we have

$$|(\mathfrak{B}\varphi)(t)-(\mathfrak{B}\psi)(t)|\leq|\varphi(t)-\psi(t)|\left[1-\frac{\varepsilon^2}{16}\right]\leq\|\varphi-\psi\|\left[1-\frac{\varepsilon^2}{16}\right].$$

(b) Suppose that for some t we have

$$|\varphi(t)-\psi(t)|\leq\varepsilon/2.$$

Then

$$|(\mathfrak{B}\varphi)(t)-(\mathfrak{B}\psi)(t)|\leq|\varphi(t)-\psi(t)|\leq(1/2)\|\varphi-\psi\|.$$

Consequently, we obtain

$$\begin{aligned}\|\mathfrak{B}\varphi-\mathfrak{B}\psi\| &\leq \max\left[1/2, 1-\frac{\varepsilon^2}{16}\right]\|\varphi-\psi\| \\ &\leq \left(1-\frac{\varepsilon^2}{16}\right)\|\varphi-\psi\|.\end{aligned}$$

□

We shall prove that the mapping H has a fixed point which solves (1.1), whenever its derivative exists.

Lemma 2. *For A defined in* (3.2)*, suppose that* (2.1)−(2.6) *and* (3.3)−(3.6) *hold. Then $A : M \to M$ is continuous in the supremum norm and maps M into a compact subset of M.*

Proof. Clearly, if φ is continuous then $A\varphi$ is also continuous. A change of variable in (3.2) shows that $(A\varphi)(t+T)=\varphi(t)$. Observe that

$$|Q(x)|\leq k_1|x|+|Q(0)|, \ \left|Q'(x)\right|\leq k_2|x|+\left|Q'(0)\right|,$$

and

$$|G(t,x,y)|\leq|G(t,x,y)-G(t,0)|+|G(t,0,0)|\leq k_3|x|+k_4|y|+|G(t,0,0)|.$$

So, for any $\varphi\in M$, we have

$$\begin{aligned}&|(A\varphi)(t)| \\ &\leq \left|\frac{c(t)}{(1-g'(t))}\right| |Q(\varphi(t-g(t)))| + \left(1-e^{-\int_{t-T}^{t}a(s)ds}\right)^{-1} \\ &\times \int_{t-T}^{t}\left[|G(u,\varphi^3(u),\varphi^3(u-g(u)))|+|h(u)|\,|Q(\varphi(u-g(u)))|\right]e^{-\int_u^t a(s)ds}du \\ &\leq \alpha(k_1L+|Q(0)|)+\left(1-e^{-\int_{t-T}^{t}a(s)ds}\right)^{-1} \\ &\times \int_{t-T}^{t}((k_3+k_4)L^3+|G(u,0,0)|+|h(u)|(k_1L+|Q(0)|))e^{-\int_u^t a(s)ds}du \\ &\leq \alpha\left(k_1+\frac{|Q(0)|}{L}\right)L+\left(1-e^{-\int_{t-T}^{t}a(s)ds}\right)^{-1}(\beta+\delta)L\int_{t-T}^{t}a(u)e^{-\int_u^t a(s)ds}du \\ &\leq \left[\alpha\left(k_1+\frac{|Q(0)|}{L}\right)+\beta+\delta\right]L\leq\frac{L}{J}<L.\end{aligned}$$

That is $A\varphi \in M$.

We show that A is continuous in the supremum norm. Let $\varphi, \psi \in M$, and let

$$\gamma = \max_{t\in[0,T]} \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}, \ \theta = \max_{u\in[t-T,t]} e^{-\int_{u}^{t} a(s)ds}, \ \sigma = \max_{t\in[0,T]} \{a(t)\},$$
$$\rho = \max_{t\in[0,T]} |G(t,0,0)|, \ \mu = \max_{t\in[0,T]} \left|\frac{c'(t)}{(1-g'(t))}\right|, \ \vartheta = \max_{t\in[0,T]} \left|\frac{g''(t)\, c(t)}{(1-g'(t))^2}\right|. \tag{3.7}$$

Then

$$\begin{aligned}
&|(A\varphi)(t) - (A\psi)(t)| \\
&\le \left|\frac{c(t)Q(\varphi(t-g(t)))}{(1-g'(t))} - \frac{c(t)Q(\psi(t-g(t)))}{(1-g'(t))}\right| + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \\
&\times \int_{t-T}^{t} \left\{\left|G(u,\varphi^3(u),\varphi^3(u-g(u))) - G(u,\psi^3(u),\psi^3(u-g(u)))\right|\right. \\
&\left. + |h(u)|\,|Q(\varphi(u-g(u))) - Q(\psi(u-g(u)))|\right\} e^{-\int_{u}^{t} a(s)ds} du \\
&\le \alpha k_1 \|\varphi-\psi\| + (k_3+k_4)\|\varphi^3-\psi^3\| \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \int_{t-T}^{t} e^{-\int_{u}^{t} a(s)ds} du \\
&+ k_1\delta\|\varphi-\psi\| \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \int_{t-T}^{t} a(u)\, e^{-\int_{u}^{t} a(s)ds} du \\
&\le (\alpha k_1 + 3(k_3+k_4)T\gamma\theta L^2 + \delta k_1)\|\varphi-\psi\|.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \frac{\varepsilon}{K}$, with $K = \alpha k_1 + 3(k_3+k_4)T\gamma\theta L^2 + \delta k_1$, where k_1, k_3 and k_4 are given by (2.3) and (2.5). Then, for $\|\varphi-\psi\| < \eta$, we obtain

$$\|A\varphi - A\psi\| \le K\|\varphi-\psi\| < \varepsilon.$$

It is left to show that A is compact. Let $\varphi_n \in M$, where n is a positive integer. Then, as above, we see that

$$\|A\varphi_n\| \le L. \tag{3.8}$$

Moreover, a direct calculation shows that

$$\begin{aligned}
(A\varphi_n)'(t) &= \frac{c'(t)Q(\varphi_n(t-g(t))) + c(t)\varphi_n'(t-g(t))Q'(\varphi_n(t-g(t)))}{1-g'(t)} \\
&+ \frac{g''(t)c(t)Q(\varphi_n(t-g(t)))}{(1-g'(t))^2} + G(t,\varphi_n^3(t),\varphi_n^3(t-g(t))) \\
&- h(t)Q(\varphi_n(t-g(t))) - a(t)\left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}
\end{aligned}$$

$$\times \int_{t-T}^{t} [G(t, \varphi_n^3(t), \varphi_n^3(t-g(t))) - h(u)Q(\varphi_n(u-g(u)))] e^{-\int_u^t a(s)ds} du.$$

By invoking the conditions (2.3) – (2.5) , (3.3), (3.5), (3.7) and (3.8) we obtain

$$\begin{aligned}|(A\varphi_n)'(t)| &\leq \mu(k_1 L + |Q(0)|) + \alpha L'(k_2 L + |Q'(0)|) + \vartheta(k_1 L + |Q(0)|) \\ &+ (k_3 + k_4) L^3 + \rho + \delta a(t)(k_1 L + |Q(0)|) \\ &+ a(t)\gamma T\theta\left[(k_3 + k_4) L^3 + \rho + \delta a(t)(k_1 L + |Q(0)|)\right] \\ &\leq (1 + \sigma\gamma T\theta)(k_3 + k_4) L^3 + (\mu + \vartheta + (1 + \sigma\gamma T\theta)\delta\sigma) k_1 L \\ &+ \alpha L'(k_2 L + |Q'(0)|) + (1 + \sigma\gamma T\theta)\rho + (\mu + \vartheta + \delta\sigma + \sigma^2\gamma T\theta\delta)|Q(0)| \\ &\leq D,\end{aligned}$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that the subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus, A is continuous and AM is a compact set. □

Lemma 3. *Suppose* (2.1) – (2.3) , (2.5) – (2.6) *and* (3.3) – (3.6) *hold. For* A, B *defined in* (3.2) *and* (3.1)*, if* $\varphi, \psi \in M$ *are arbitrary, then*

$$A\varphi + B\psi : M \to M.$$

Moreover, B *is a large contraction on* M *with a unique fixed point in* M*.*

Proof. Let $\varphi, \psi \in M$ be arbitrary. Note first that $|\psi(t)| \leq \sqrt{3}/3$ implies

$$|\psi(t) - \psi^3(t)| \leq (2\sqrt{3})/9.$$

Using the definition of B, and the result of Lemma 2, we obtain

$$\begin{aligned}|(A\varphi)(t) + (B\psi)(t)| &\leq |(A\varphi)(t)| + |(B\psi)(t)| \\ &\leq \frac{\sqrt{3}}{3J} + \frac{2\sqrt{3}}{9} \leq L.\end{aligned}$$

Thus, $A\varphi + B\psi \in M$. It is left to show that B is a large contraction with a unique fixed point in M. Proposition 1 shows that $\psi - \psi^3$ is a large contraction in the supremum norm. For any ε, from the proof of that proposition, we have found a $\lambda < 1$, such that

$$\begin{aligned}|(B\psi)(t) - (B\varphi)(t)| &\leq \left(1 - e^{-\int_{t-T}^t a(s)ds}\right)^{-1} \int_{t-T}^{t} a(u)\lambda\|\psi - \varphi\| e^{-\int_u^t a(s)ds} du \\ &\leq \lambda\|\psi - \varphi\|.\end{aligned}$$

Further, since $0 \in M$, the above inequality shows that $B : M \to M$ when $\varphi = 0$. This completes the proof. □

Theorem 4. *Let* $(S, \|.\|)$ *be the Banach space of continuous* T*-periodic real functions and* $M = \{\varphi \in S : \|\varphi\| \leq L,\ \|\varphi'\| \leq L'\}$*, where* $L = \sqrt{3}/3$*. Suppose* (2.1) $-$ (2.6) *and* (3.3) $-$ (3.6) *hold. Then equation* (1.1) *possesses a* T*-periodic solution* φ *in the subset* M*.*

Proof. By Lemma 1, φ is a solution of (1.1) if

$$\varphi = A\varphi + B\varphi,$$

where A and B are given by (3.2), (3.1) respectively. By Lemma 2, $A : M \to M$ is continuous and AM is contained in a compact subset of M. By Lemma 3, $A\varphi + B\psi \in M$ whenever $\varphi, \psi \in M$. Moreover, $B : M \to M$ is a large contraction. Clearly, all the hypotheses of Theorem 3 of Krasnoselskii are satisfied. Thus, there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. Hence (1.1) has a T-periodic solution in M. □

Remark 1. When $Q(x) = x$, Theorem 4 reduces to Theorem 3.4 in [5].

Example 1. Let S and M be as in Theorem 4 with $T = 2\pi$ and consider the totally nonlinear neutral equation

$$\begin{aligned} x'(t) &= -6.10^{-2}x^3(t) + 10^{-4}\cos t.x'(t-2)\cos(x(t-2)) \\ &\quad + 10^{-4}\sin(t)\left(\cos t + \cos\left(x^3(t)\right) + \sin\left(x^3(t-2)\right)\right). \end{aligned} \tag{3.8}$$

Then

$$T = 2\pi,\ a(t) = 6.10^{-2},\ c(t) = 10^{-4}\cos t,\ Q(x) = \sin x,$$
$$G(t, x, y) = 10^{-4}\sin(t)(\cos t + \cos(x) + \sin(y)),\ g(t) = 2.$$

Doing straightforward computations, we obtain

$$k_1 = k_2 = 1, \quad k_3 = k_4 = 10^{-4}, \quad \alpha = \mu = 10^{-4}, \quad Q(0) = 0,$$
$$Q'(0) = 1, \quad \rho = 2.10^{-4}, \quad \vartheta = 0, \quad \gamma = (1 - e^{-0.12\pi})^{-1}, \quad \theta = 1.$$

By substituting $\beta = \delta = 10^{-2}$ in (3.3) and (3.4) we have that any $J \in [3, 49]$ satisfies (3.6). All hypotheses of Theorem 4 are fulfilled and so the equation (3.9) have at least a 2π-periodic solution belonging to M.

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Abdelouaheb Ardjouni and Ahcéne Djoudi
Department of Mathematics
Faculty of Sciences
University of Annaba
P.O. Box 12
Annaba, Algeria
E–mail: abd_ardjouni@yahoo.fr
E–mail: adjoudi@yahoo.com