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KIEPERT TRIANGLES IN AN ISOTROPIC PLANE

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Abstract. In this paper the concept of the Kiepert triangle of an allowable triangle in an isotropic plane is introduced. The relationships between the areas and the Brocard angles of the standard triangle and its Kiepert triangle are studied. It is also proved that an allowable triangle and any of its Kiepert triangles are homologic. In the case of a standard triangle the expressions for the center and the axis of this homology are given.

In the Euclidean geometry of triangles the Kiepert hyperbola and the Kiepert parabola of a given triangle play a significant role (see eg. [1]). The first conic is a circumscribed rectangular hyperbola of that triangle, and the second one is inscribed in this triangle. Both conics can be obtained in different ways, but the most suitable way is by means of the so called Kiepert triangles of the considered triangle.

Let ABC be a given triangle and φ a given angle. If BCA', CAB', ABC' are mutually similar isosceles triangles constructed on the bases BC, CA, AB with the base angle φ, then $A'B'C'$ is a Kiepert triangle of the given triangle ABC. It is homologic with the triangle ABC, i.e. the lines AA', BB', CC' pass through one point T, and the points $BC \cap B'C'$, $CA \cap C'A'$, $AB \cap A'B'$ lie on one line $\mathcal{T}$ (*Figure 1*). With the variable angle $\varphi \in (-\pi, \pi)$ the point T describes the Kiepert hyperbola, and the line $\mathcal{T}$ envelopes the Kiepert parabola of the triangle ABC. There are two values of the angle φ, for which the points A', B', C' lie on the line $\mathcal{T}$. The two obtained lines $\mathcal{T}$ are the so called Steiner axes of the triangle ABC.

In the isotropic geometry the studying of the conics analogous to the Kiepert hyperbola and the Kiepert parabola requires firstly a proper knowledge of the properties of the analogues of the Kiepert triangles in this geometry, and which is the intention of this paper.

The authors are going to continue the research and in this way widen and apply the obtained results to the investigation of the Kiepert parabola and hyperbola in the isotropic plane.

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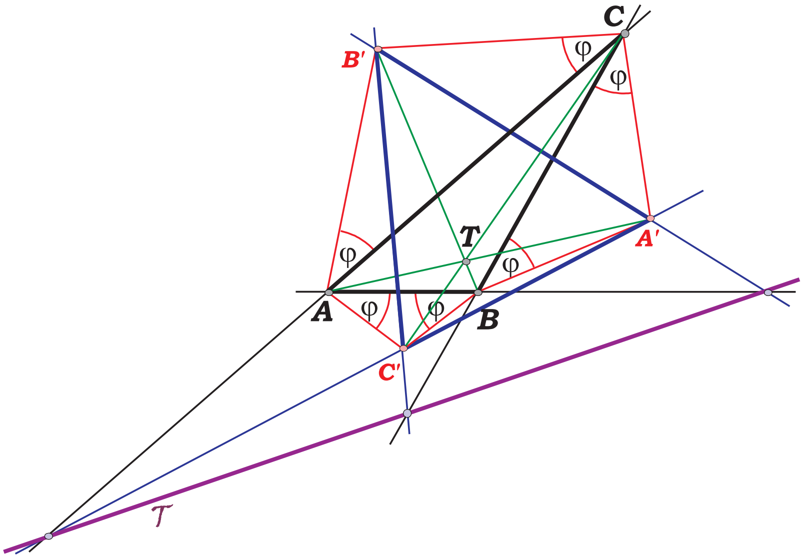

Figure 1.

In an *isotropic plane* (see e.g. [6] and [7]) the *distance* between the two points $P_i = (x_i, y_i)$ $(i = 1, 2)$ is defined by $P_1P_2 = x_2 - x_1$ and the two lines with the equations $y = k_i x + l_i$ $(i = 1, 2)$ form the *angle* $k_2 - k_1$. Two points P_1, P_2 with $x_1 = x_2$ are said to be *parallel*, and we shall say they are on the same *isotropic* line. Any isotropic line is perpendicular to any nonisotropic line. Two lines with $k_1 = k_2$ are *parallel*. For the two parallel points P_1, P_2 their span is defined by $s(P_1, P_2) = y_2 - y_1$.

A triangle is said to be *allowable* if none of its sides are isotropic. Each allowable triangle ABC can be set by a suitable choice of the coordinate system in the *standard position*, in which its circumscribed circle has the equation $y = x^2$, its vertices are the points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{1}$$

and its sides BC, CA, AB have the equations

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab \tag{2}$$

where

$$a + b + c = 0. \tag{3}$$

Then we shall say that ABC is the *standard triangle*. To prove the geometric facts for each allowable triangle it is sufficient to give a proof for the standard triangle (see [4]).

With the labels

$$p = abc, \quad q = bc + ca + ab \tag{4}$$

a number of useful equalities are proved in [4] as for example $a^2+b^2+c^2=-2q$, $a^2=bc-q$, $b^2+bc+c^2=-q$, $2q-3ab=(b-c)(c-a)$, $q+3bc=-(b-c)^2$.

Now, we will define the concept of the Kiepert triangle of the allowable triangle ABC.

If A_m, B_m, C_m are the midpoints of the sides of the allowable triangle ABC and if A', B', C' are the points on the perpendicular bisectors of these sides such that the spans A_mA', B_mB', C_mC' are proportional to the lengths of the sides BC, CA, AB, then the triangle $A'B'C'$ is the so called *Kiepert triangle* of the triangle ABC (*Figure 2*).

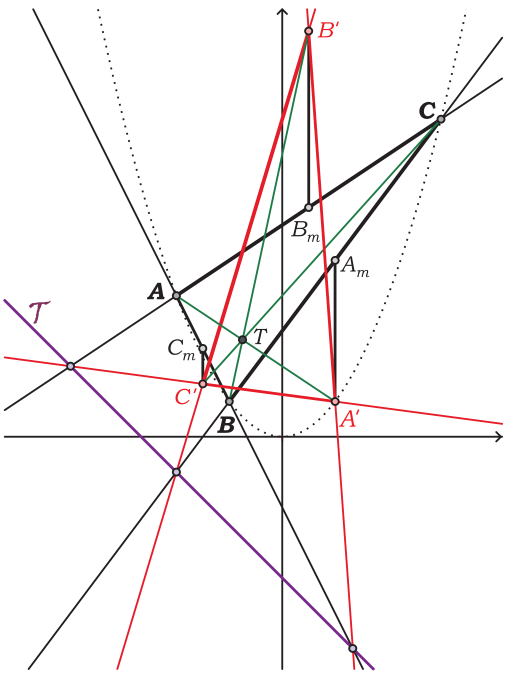

Figure 2.

Theorem 1. *Any Kiepert triangle $A'B'C'$ of the standard triangle ABC has the vertices*

$$A'=\left(-\frac{1}{2}a,\ -\frac{1}{2}(q+bc)+\frac{1}{2}(b-c)t\right),$$

$$B'=\left(-\frac{1}{2}b,-\frac{1}{2}(q+ca)+\frac{1}{2}(c-a)t\right),$$

$$C'=\left(-\frac{1}{2}c,-\frac{1}{2}(q+ab)+\frac{1}{2}(a-b)t\right), \tag{5}$$

where t is a real parameter. Its side $B'C'$ has the equation

$$y=\left(\frac{3a}{b-c}t-a\right)x+\frac{q}{b-c}t+\frac{1}{2}bc-q, \tag{6}$$

and the equations of the sides $C'A'$ and $A'B'$ are analogous to (6).

Proof. The midpoint of the points $B=(b,b^2)$ and $C=(c,c^2)$ is

$$A_m=\left(-\frac{1}{2}a,-\frac{1}{2}(q+bc)\right),$$

due to $b+c=-a$ and $b^2+c^2=-(q+bc)$. Then, because of $BC=c-b$ we get the first of the three analogous equalities (5). The points B' and C' from (5) lie on the line with the equation (6) as, for example for the point B' we get the following

$$\begin{aligned}\left(\frac{3a}{b-c}t-a\right)\left(-\frac{1}{2}b\right)+\frac{q}{b-c}t+\frac{1}{2}bc-q&=\frac{t}{2(b-c)}(2q-3ab)-\frac{1}{2}b^2-q\\&=\frac{t}{2(b-c)}(b-c)(c-a)-\frac{1}{2}(ca-q)-q\\&=-\frac{1}{2}(q+ca)+\frac{1}{2}(c-a)t.\end{aligned}$$

□

It can also be proved that all Kiepert triangles of the triangle ABC have the centroid in the centroid $G=(0,-\frac{2}{3}q)$ of the triangle ABC because of

$$-\frac{1}{6}(3q+bc+ca+ab)=-\frac{2}{3}q.$$

We shall say that the Kiepert triangle with the vertices (1) has the *parameter t*. It is obvious that the complementary triangle $A_mB_mC_m$ of the triangle ABC has the parameter $t=0$.

In the following we will investigate the relationship between the areas and the Brocard angles of the standard triangle and its Kiepert triangle.

Theorem 2. *Let $A'B'C'$ be the Kiepert triangle with the parameter t of the standard triangle ABC. If $\triangle$ and $\triangle'$ are the areas, ω and ω' the Brocard angles of the triangles ABC and $A'B'C'$, respectively, then the following equalities are valid:*

$$\triangle'=\frac{1}{4}\triangle-\frac{3}{4}qt=-\frac{3}{8}q(\omega+2t),$$

$$\omega'=\omega+2t.$$

Proof. In [3] the equalities

$$2\triangle = (b-c)(c-a)(a-b), \qquad BC^2 + CA^2 + AB^2 = -6q,$$

$$\omega = \frac{4\triangle}{BC^2 + CA^2 + AB^2} = -\frac{1}{3q}(b-c)(c-a)(a-b) = -\frac{2\triangle}{3q}$$

are proved. For the points (5) we get e.g.

$$y_{B'} - y_{C'} = \frac{1}{2}a(b-c) + \frac{1}{2}(b+c-2a)t = \frac{1}{2}a(b-c) - \frac{3}{2}at,$$

$$x_{A'}(y_{B'} - y_{C'}) = -\frac{1}{4}a^2(b-c) + \frac{3}{4}a^2 t$$

and therefore

$$\begin{aligned} 2\triangle' &= x_{A'}(y_{B'} - y_{C'}) + x_{B'}(y_{C'} - y_{A'}) + x_{C'}(y_{A'} - y_{B'}) \\ &= -\frac{1}{4}[a^2(b-c) + b^2(c-a) + c^2(a-b)] + \frac{3}{4}(a^2+b^2+c^2)t \\ &= \frac{1}{4}(b-c)(c-a)(a-b) - \frac{3}{2}qt = \frac{1}{4}(2\triangle - 6qt) \\ &= \frac{1}{4}(-3q\omega - 6qt) = -\frac{3}{4}q(\omega + 2t). \end{aligned}$$

It is obvious

$$B'C'^2 + C'A'^2 + A'B'^2 = \frac{1}{4}(BC^2 + CA^2 + AB^2) = -\frac{3}{2}q,$$

and then it follows

$$\omega' = \frac{4\triangle'}{B'C'^2 + C'A'^2 + A'B'^2} = \omega + 2t.$$

□

Corollary 1. *The points A', B', C' from Theorem* 1 *are collinear if and only if $t = -\frac{1}{2}\omega$.*

Theorem 3. *The allowable triangle ABC and each of its Kiepert triangles $A'B'C'$ are homologic, i.e. the lines AA', BB', CC' pass through one point T, and the points $BC \cap B'C'$, $CA \cap C'A'$, $AB \cap A'B'$ lie on one line $\mathcal{T}$ (Figure 2). In the case of the standard triangle ABC the point*

$$T = \left(\frac{3pt}{q(2t+3\omega)}, -\frac{3\omega t^2 + 2qt + 6q\omega}{3(2t+3\omega)} \right), \tag{7}$$

is the center of homology and the axis of homology $\mathcal{T}$ has the equation

$$y = \frac{6pt}{q(2t+3\omega)}x + \frac{q}{6t}(2t+3\omega), \tag{8}$$

where ω is the Brocard angle of the triangle ABC.

Proof. The line with the equation

$$3ay = [3bc - q - (b - c)t]x + a(b - c)t - 2aq \tag{9}$$

passes through the points $A = (a, a^2)$ and A' from (5) because we get

$$[3bc - q - (b - c)t]a + a(b - c)t - 2aq = 3a(bc - q) = 3a \cdot a^2,$$

$$[3bc - q - (b - c)t]\left(-\frac{1}{2}a\right) + a(b - c)t - 2aq = 3a\left[-\frac{1}{2}q - \frac{1}{2}bc + \frac{1}{2}(b - c)t\right],$$

so this is the line AA'. It passes through the point T from (7) as, by the substitution of the abscissa of that point in the equation (9) we get the ordinate (without the factor a)

$$\begin{aligned} 3q(2t + 3\omega)y &= [3bc - q - (b - c)t] \cdot 3bct + [(b - c)t - 2q] \cdot q(2t + 3\omega) \\ &= (b - c)(2q - 3bc)t^2 + [9b^2c^2 - 3bcq - 4q^2 + 3(b - c)q\omega]t - 6q^2\omega \\ &= -3q\omega t^2 - 2q^2t - 6q^2\omega \end{aligned}$$

because of the equalities

$$(b - c)(2q - 3bc) = (b - c)(c - a)(a - b) = -3q\omega,$$

$$3(b-c)q\omega = -(b-c)^2(c-a)(a-b) = (q+3bc)(2q-3bc) = 2q^2+3bcq-9b^2c^2.$$

□

In the case of Corollary 1 the points A', B', C' lie on the line $\mathcal{T}$, whose equation (8) in that case gets the form

$$y = -\frac{3p}{2q}x - \frac{2}{3}q.$$

By the analogy with the Euclidean case this line will be called the *Steiner axis* of the triangle ABC.

It could be interesting to try to answer the following question. Is it possible that the point T from (7) coincides with any vertices of the triangle ABC?

Making the abscissa of the points T and A equal we get the equation $3bct = 2qt + 3q\omega$, i.e. $(2q - 3bc)t = -3q\omega$ without the factor a, or the equation $(c - a)(a - b)t = (b - c)(c - a)(a - b)$ with the solution $t = b - c$. With this solution the ordinate of the point T is equal to a^2 since we get

$$\begin{aligned} 3\omega t^2 + 2qt + 6q\omega + 3a^2(2t + 3\omega) &= 2t(q + 3a^2) + 3\omega[(b - c)^2 + 2q + 3a^2] \\ &= 2t(3bc - 2q) + 3\omega[a^2 + b^2 + c^2 + 2(q + a^2 - bc)] \\ &= -2(b - c)(c - a)(a - b) + 3\omega(-2q) \\ &= 6q\omega - 6q\omega = 0. \end{aligned}$$

Thus, we have just proved the following statement:

Theorem 4. *Let $A'B'C'$ be the Kiepert triangle with the parameter t of the standard triangle ABC and let T be the center of homology of the triangles ABC and $A'B'C'$. Then $T = A$, $T = B$ or $T = C$ according to whether $t = b - c$, $t = c - a$ respectively $t = a - b$.*

Theorem 5. *Let T be the center of homology of the standard triangle ABC with the Brocard angle ω and its Kiepert triangle with the parameter t. If $t = -3\omega$, then the point T is of the form $T = \left(\frac{3p}{q}, 3\omega^2\right)$ and it lies on the circumscribed Steiner ellipse of the triangle ABC.*

Proof. The first statement immediately follows from (7) where $t = -3\omega$. According to [8] the circumscribed Steiner ellipse of the triangle ABC has the equation

$$q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 = 0$$

and the obtained point T lies on it, because of

$$9p^2 - \frac{81p^2}{q}\omega^2 - 27q\omega^4 - 18p^2 - 12q^2\omega^2 + 9p^2 = -\frac{3\omega^2}{q}(27p^2 + 9q^2\omega^2 + 4q^3) = 0,$$

where we have used the equality

$$\begin{aligned} 9q^2\omega^2 &= (b-c)^2(c-a)^2(a-b)^2 = -(q+3bc)(q+3ca)(q+3ab) \\ &= -q^3 - 3q^2(bc+ca+ab) - 3qabc(a+b+c) - 27a^2b^2c^2 \\ &= -4q^3 - 27p^2. \end{aligned}$$

□

Theorem 6. *The point*

$$T' = \left(\frac{3p}{2q}, \frac{1}{2}\omega t - \frac{1}{3}q\right) \tag{10}$$

is isogonal to the point T from Theorem 3 with regard to the triangle ABC.

Proof. With $x' = \frac{3p}{2q}$, $y' = \frac{1}{2}\omega t - \frac{1}{3}q$ we get

$$\begin{aligned} y' - x'^2 &= \frac{1}{2}\omega t - \frac{1}{3}q - \frac{9p^2}{4q^2} = \frac{1}{12q^2}(6q^2\omega t - 27p^2 - 4q^3) \\ &= \frac{1}{12q^2}(6q^2\omega t + 9q^2\omega^2) = \frac{1}{4}\omega(27 + 3\omega), \end{aligned}$$

$$x'y' + qx' - p = \frac{3p}{2q}\left(\frac{1}{2}\omega t + \frac{2}{3}q\right) - p = \frac{3p}{4q}\omega t,$$

$$\begin{aligned}px' - qy' - y'^2 &= \frac{3p^2}{2q} - \left(\frac{1}{2}\omega t - \frac{1}{3}q\right)\left(\frac{1}{2}\omega t + \frac{2}{3}q\right)\\ &= -\frac{1}{4}\omega^2 t^2 - \frac{1}{6}q\omega t + \frac{1}{18q}(27p^2 + 4q^3)\\ &= -\frac{1}{4}\omega^2 t^2 - \frac{1}{6}q\omega t + \frac{1}{18q}(-9q^2\omega^2) = -\frac{\omega}{12}(3\omega t^2 + 2qt + 6q\omega)\end{aligned}$$

since according to the proof of the previous theorem the equality $27p^2+4q^3 = -9q^2\omega^2$ is valid. According to [5] the point isogonal to the point $T' = (x', y')$ has the coordinates

$$x = \frac{x'y' + qx' - p}{y' - x'^2} = \frac{3pt}{q(2t + 3\omega)},$$

$$y = \frac{px' - qy' - y'^2}{y' - x'^2} = -\frac{3\omega t^2 + 2qt + 6q\omega}{3(2t + 3\omega)},$$

and it is the point T from formula (7). □

According to [2] the Brocard diameter of the standard triangle ABC has the equation $x = \frac{3p}{2q}$. Using the above fact a number of statements can be proved as follows.

Corollary 2. *The point T', which is isogonal to the center T of the homology of this triangle and any of its Kiepert triangles with regard to the allowable triangle ABC, lies on the Brocard diameter of the triangle ABC.*

Theorem 7. *If the points T_1, T_2 are the centers of homologies of the standard triangle ABC with the Brocard angle ω and its Kiepert triangles with the parameters t_1, t_2, then the line T_1T_2 has the equation*

$$y = -\frac{q}{9p}(2t_1t_2 + 3\omega t_1 + 3\omega t_2 - 2q)x + \frac{1}{3}(t_1t_2 - 2q). \tag{11}$$

Proof. The point T from (7) with $t = t_1$ lies on the line (11) because of

$$\begin{aligned}&-\frac{q}{9p}(2t_1t_2 + 3\omega t_1 + 3\omega t_2 - 2q)\cdot\frac{3pt_1}{q(2t_1 + 3\omega)} + \frac{1}{3}(t_1t_2 - 2q)\\ &= -\frac{1}{3(2t_1 + 3\omega)}[(2t_1t_2 + 3\omega t_1 + 3\omega t_2 - 2q)t_1 - (t_1t_2 - 2q)(2t_1 + 3\omega)]\\ &= -\frac{1}{3(2t_1 + 3\omega)}(3\omega t_1^2 + 2qt_1 + 6q\omega).\end{aligned}$$

□

Theorem 8. *If the points T_1, T_2, T_3 are the centers of homologies of the standard triangle ABC and its Kiepert triangles with the parameters t_1, t_2,*

t_3 and if the point T_3' is isogonal to the point T_3 with regard to the triangle ABC, then the points T_1, T_2, T_3' are collinear if and only if

$$t_1 + t_2 + t_3 = 0. \tag{12}$$

Proof. The point T' from (10) with $t = t_3$ lies on the line T_1T_2 with the equation (11) under the condition

$$\frac{1}{2}\omega t_3 - \frac{1}{3}q = -\frac{q}{9p}(2t_1t_2 + 3\omega t_1 + 3\omega t_2 - 2q)\frac{3p}{2q} + \frac{1}{3}(t_1t_2 - 2q)$$

which, after rearrangement, obtains the form $\frac{1}{2}\omega t_3 = -\frac{1}{2}\omega(t_1 + t_2)$, i.e. the form (12). □

Theorem 8 and Corollary 2 immediately imply:

Corollary 3. *Let T_1, T_2, T_3 be the centers of homologies of an allowable triangle ABC and its three Kiepert triangles, and let T_1', T_2', T_3' be the points isogonal to the points T_1, T_2, T_3 with regard to the triangle ABC. If the points T_1, T_2, T_3' are collinear, then the triples of the points T_1, T_2', T_3 and T_1', T_2, T_3 are collinear too, i.e. the points T_1, T_1'; T_2, T_2'; T_3, T_3' are the pairs of opposite vertices of one complete quadrilateral.*

It might be interesting to find the answer to the following question. Is it possible that the line $\mathcal{T}$ having equation (8) coincides with a side of the triangle ABC?

The equation of the side BC from (2) coincides with the equation (8) under the condition $6bct + q(2t + 3\omega) = 0$, i.e. $2t(q + 3bc) = -3q\omega$ or the equation $-2t(b-c)^2 = (b-c)(c-a)(a-b)$, with the solution $t = -\frac{(c-a)(a-b)}{2(b-c)}$. Thus, we have proved:

Theorem 9. *Let $A'B'C'$ be the Kiepert triangle with the parameter t of the standard triangle ABC and let $\mathcal{T}$ be the axis of the homology of the triangles ABC and $A'B'C'$. Then $\mathcal{T}$ is the line BC if and only if $t = -\frac{(c-a)(a-b)}{2(b-c)}$, and similarly it is also valid for the lines CA and AB.*

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